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Compatible Geometric Matchings*

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§1 Introduction

A *geometric graph* is a simple graph G, where the vertex-set $V(G)$ is a finite set of points in the plane, and each edge in $E(G)$ is a closed segment whose endpoints belong to $V(G)$. We assume that no three vertices are collinear. A geometric graph is *non-crossing* if no two edges cross. Two non-crossing geometric graphs are *compatible* if they have the same vertex set and their union is non-crossing. A *matching* is a non-crossing geometric graph in which every vertex has degree at most one. A matching is *perfect* if every vertex has degree exactly one. We say that a (perfect) matching is a (*perfect*) *matching of* its vertex set. Our focus is on compatible perfect matchings.

First consider the problem of transforming a given perfect matching into another given perfect matching on the same vertex set. Let S be a set of n points in the plane, with n even. For perfect matchings M and M' of S, a *transformation between M and M' of length k* is a sequence $M = M_0, M_1, \ldots, M_k = M'$ of perfect matchings of S, such that each M_i is compatible with M_{i+1}. Houle et al. [4] proved that there is a transformation of length $n - 2$ between any given pair of perfect matchings of S. We improve this bound from linear to logarithmic. The proof is omitted.

Theorem 1. *For every set S of $2n$ points in general position, there is a transformation of length $2\lceil \log_2 n \rceil$ between any given pair of perfect matchings of S.*

The remaining results concern the following conjecture. Two geometric graphs are *disjoint* if they have no edge in common. A (multi)graph is *even* or *odd* if the number of edges is even or odd.

Compatible Matching Conjecture. For every even perfect matching M, there is a perfect matching that is disjoint and compatible with M.

Note that the assumption that the given perfect matching is even is necessary, since for example, an odd number of parallel chords of a circle have no disjoint compatible perfect matching. §2 describes progress toward the proof of this conjecture. In particular, we introduce a sequence of stronger conjectures that imply it. Then in §3 we establish the Compatible Matching Conjecture for the following two special cases: perfect matchings that consist of vertical and horizontal segments, and perfect matchings that arise from convex-hull-connected sets of segments. Finally in §4, we prove two relaxations of the Compatible Matching Conjecture. First, we prove that every perfect matching with n edges has a disjoint compatible (partial) matching with approximately $4n/5$ edges. Second, we prove that every perfect matching has a disjoint perfect matching possibly with crossings.

§2 Extensions

We attack the Compatible Matching Conjecture using extensions. Let M be a perfect matching with n edges. An *extension* of M is a set of segments and rays obtained by extending each segment in M, in some given order, by rays in both directions. Each ray is extended until it hits another segment, or a previous extension, or the ray goes to infinity. An extension L of M defines a convex subdivision with $n + 1$ cells. Each endpoint of M is on the boundary of exactly two cells.

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Our approach is to first assign each vertex of M to one of its two neighbouring cells, and then compute a perfect matching of the vertices assigned to each cell. This assignment is modelled by an orientation of the edges of the *dual multigraph* G. The vertices of G are the cells of the subdivision. For every endpoint v of M, add an edge to G between the vertices that correspond to the two cells of which v is on the boundary. Thus G has $n+1$ vertices and $2n$ edges.

An orientation of a multigraph is *even* if every vertex has even indegree. Frank et al. [3] proved that a multigraph admits an even orientation if and only if each component is even.

Extension Conjecture. Every even perfect matching M has an extension L, such that the associated dual multigraph G admits an even orientation, with the property that whenever a vertex v of G has indegree 2, the two incoming edges at v do not arise from the same segment in M.

Lemma 2. *The Extension Conjecture implies the Compatible Matching Conjecture.*

Proof. For each oriented edge xy of G corresponding to a vertex v of M, *assign* v to the cell y. Since the orientation of G is even, an even number of vertices are assigned to each cell. Moreover, by the final assumption in the Extension Conjecture, if exactly two vertices are assigned to one cell, then the two vertices are not adjacent in M. It follows that M restricted to the set of vertices that are assigned to each cell has a disjoint compatible perfect matching (contained within the cell). Their union is a perfect matching of the whole set that is disjoint and compatible with M. □

Two Subgraphs Conjecture. Every even perfect matching M has an extension L, such that the associated dual graph G has an edge-partition into two subgraphs G_1 and G_2, such that each component of G_1 is even, each component of G_2 is even, and for every segment vw of M, the edge of G corresponding to v is in a different subgraph from the edge of G corresponding to w.

Lemma 3. *The Two Subgraphs Conjecture implies the Extension Conjecture.*

Proof. Since each component of G_1 and G_2 is even, each of G_1 and G_2 admit an even orientation [3], the union of which is an even orientation of G, such that if a vertex has indegree 2, then the two incoming edges are both in G_1 or both in G_2, and thus arise from distinct segments. Hence the even orientation satisfies the Extension Conjecture. □

Two Trees Conjecture. Every (even or odd) perfect matching M has an extension L, such that the associated dual graph G has an edge-partition into two spanning trees T_1 and T_2, and for every segment vw of M, the edge of G corresponding to v is in a different tree from the edge of G corresponding to w.

Lemma 4. *The Two Trees Conjecture implies the Extension Conjecture.*

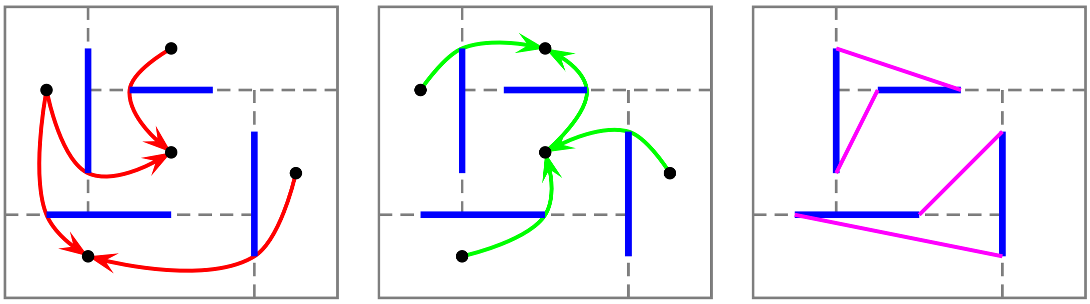
Proof. We may assume that M is even. Thus G has an odd number of vertices, and each T_i has an even number of edges. Hence each T_i has an even orientation. Their union is an even orientation of G, such that if a vertex of G has indegree 2, then the two incoming edges are both in T_1 or both in T_2, and thus arise from distinct segments. Hence the Extension Conjecture is satisfied. □

Note that Benbernou et al. [2] made some progress on the Two Trees Conjecture. They proved that every perfect matching has an extension such that the associated dual multigraph G is 2-edge-connected, which is a necessary condition for G to have the desired partition into two trees.

§3 Two Special Cases

Theorem 5. *Every even perfect matching M consisting of vertical and horizontal segments has a disjoint compatible perfect matching.*

Proof Sketch. As illustrated in the next figure, first extend each horizontal segment, and then extend each vertical segment. For each horizontal segment, colour the edge of G through the left endpoint *red* and colour the edge of G through the right endpoint *green*. For each vertical segment, colour the edge of G through the bottom endpoint *red* and colour the edge of G through the top endpoint *green*. We prove that the red and green subgraphs are both spanning trees of G. For every segment vw, the edge of G passing through v is in a different tree from the edge of G passing through w. Thus the Two Trees Conjecture is satisfied. The result follows from Lemmas 2, 3, and 4. □



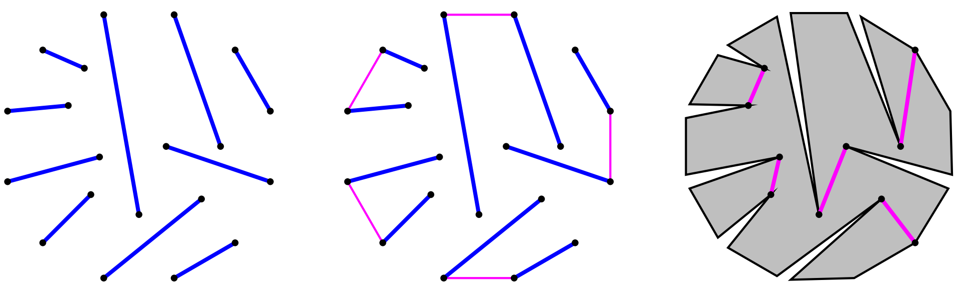
Lemma 6 (Abellanas et al. [1])**.** *Let R be the set of reflex vertices of a simple polygon P, and let S be any finite set of points on the boundary of P or in its interior, such that $R \subseteq S$ and $|S|$ is even. Then there is a perfect matching M of S such that every segment in M is inside the polygon P.*

A set M of pairwise disjoint segments is *convex-hull-connected* if each segment has at least one endpoint on the boundary of the convex hull of M.

Theorem 7. *For every even convex-hull-connected perfect matching M, there is a perfect matching that is disjoint and compatible with M.*

Proof. We proceed by induction on the number of segments in M. A segment vw in M is a *splitter* if v and w are non-consecutive points on the boundary of the convex hull (amongst the set of vertices of M). First suppose that there is a splitter segment vw in M. Of the sets of segments on the two sides of vw, one has an even non-zero number of segments, and the other has an odd number of segments. Group vw with the odd-sized set. Thus M is now partitioned into two even convex-hull-connected perfect matchings M_1 and M_2. By induction, there is a perfect matching M_1' that is disjoint and compatible with M_1, and there is a perfect matching M_2' that is disjoint and compatible with M_2. Hence $M_1' \cup M_2'$ is a perfect matching that is disjoint and compatible with M.

Now assume M has no splitter, as illustrated in the next figure. A *gap* is an edge of the convex hull of M that is not a segment in M. Since M is even and there are no splitter segments, the number of gaps is even. Let B be a set of alternate gaps on the convex hull. Thus B forms a set of segments, such that for every segment xy in M, exactly one of x and y is an endpoint of a segment in B. For each segment xy of M with exactly one vertex, say x, on the convex hull, let $W(xy)$ be an infinitesimally thick wedge centred at y containing xy. Let P be the polygon obtained from the convex hull of M by removing each $W(xy)$. Thus every reflex vertex of P is an endpoint of a segment in M not intersecting B. Since M is even and B includes exactly one endpoint from each segment in M, the number of endpoints of segments in M not intersecting B is even. By Lemma 6, there is a perfect matching Q of the set of endpoints of segments in M not intersecting B, such that every segment in Q is inside P. Since every segment in B is on the boundary of the convex hull, $B \cup Q$ is a perfect matching that is disjoint and compatible with M. □



§4 Two Relaxations

Given that the Compatible Matching Conjecture has remained elusive, it is natural to consider how large a disjoint compatible matching can be guaranteed.

Theorem 8. *Let S be a set of $2n$ points in the plane in general position, with n even, and let M be a perfect matching of S. Then there is a matching M' of S with at least $\frac{1}{5}(4n-1)$ segments, such that M and M' are compatible and disjoint.*

Proof. Without loss of generality, no segment in M is vertical. Fix a bounding box around the segments. First extend each segment to the right. Then extend each segment to the left. We obtain a convex subdivision with $n+1$ faces. Let G be the corresponding dual graph. So G has $n+1$ vertices. Colour each edge of G that corresponds to a right endpoint *red*. Colour each edge of G that corresponds to a left endpoint *blue*. Let R and B be the spanning subgraphs of G respectively consisting of the red and blue edges. Each of B and R have $n+1$ vertices and n edges.

By considering the change in the dual graph after each extension, we prove that B is a spanning tree of G. For each odd component X of R, there is an edge e in X, such that $X-e$ has no odd component. Delete e from R. We are left with a subgraph R' of R with no odd component. Since n is even, the one component of B is even. By construction, for every segment vw of M, the edge of G corresponding to v is coloured differently from the edge of G corresponding to w. Hence $B \cup R'$ satisfies the Two Subgraphs Conjecture. By Lemmas 2 and 3, there is a partial matching M' of S that is compatible and disjoint with M, and the number of segments in M' equals half the number of edges in $B \cup R'$, which is $2n$ minus the number of odd components in R.

The following lemma is easily proved: *Every planar graph with n vertices and m edges has at most $\frac{1}{5}(3n-m)$ odd components, with equality only if every component is K_2.* Thus R (which has $n+1$ vertices, n edges, and thus has some non-K_2 component) has at most $\frac{1}{5}(3(n+1)-n-1) = \frac{2}{5}(n+1)$ odd components. Hence M' has at least $\frac{1}{2}(2n - \frac{2}{5}(n+1)) = \frac{1}{5}(4n-1)$ segments. □

Now we prove a relaxation of the Compatible Matching Conjecture that allows some crossings.

Theorem 9. *Let M be an even perfect matching with no vertical segment. Let L be the set of left endpoints of M, and let R be the set of right endpoints of M. Then there is a perfect matching M_L of L, and a perfect matching M_R of R, such that no edge in M crosses an edge in $M_L \cup M_R$ (but an edge in M_L might cross an edge in M_R).*

Proof Sketch. Let C be a convex polygon bounding M. In a similar fashion to the proof of Theorem 7, extend each segment of M by an infinitesimally thickened ray from its left endpoint. Removing the thickened rays from the interior of C, we obtain a polygon whose reflex vertices are the right endpoints of the segments in M. Since M is even, by Lemma 6 with $R = S$, there is a perfect matching M_R of R such that $M_R \cup M$ is non-crossing. M_L is obtained similarly. □

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