Forschungsschwerpunkt S92

Industrial Geometry

http://www.ig.jku.at

INDUSTRIAL GEOMETRY

FSP Report No. 78

A Quadratic Clipping Step with Superquadratic Convergence for bivariate Polynomial Systems

Bert Jüttler, Brian Moore

October 2008

FWF Der Wissenschaftsfonds.

A Quadratic Clipping Step with Superquadratic Convergence for Bivariate Polynomial Systems

Bert Jüttler and Brian Moore

Abstract. A new numerical approach to compute all real roots of a system of two bivariate polynomial equations over a given box is described. Using the Bernstein-Bézier representation, we compute the best linear approximant and the best quadratic approximant of the two polynomials with respect to the L^2 norm. Using these approximations and bounds on the approximation errors, we obtain a fat line bounding the zero set first of the first polynomial and a fat conic bounding the zero set of the second polynomial. By intersecting these fat curves, which requires solely the solution of linear and quadratic equations, we derive a reduced subbox enclosing the roots. This algorithm is combined with splitting steps, in order to isolate the roots. It is applied iteratively until a certain accuracy is obtained. Using a suitable preprocessing step, we prove that the convergence rate is 3 for single roots. In addition, experimental results indicate that the convergence rate is superlinear (1.5) for double roots.

Mathematics Subject Classification (2000). Primary 30C15; Secondary 65H10.

Keywords. root finding, polynomial, Bézier clipping.

1. Introduction

Solving zero-dimensional systems of polynomial equations is a fundamental problem in various applications, including computer-aided design, robotics and manufacturing. Many different approaches can be used to solve such problems [4]. Symbolic solvers form an important class of such solvers. Recently, the efficient implementation of Groebner basis computation by Faugère [6, 7] combined with efficient computation of the solutions by Rouillier [17, 18] led to a dramatic increase in the efficiency of such methods.

B. Jüttler was supported through the Austrian Science Fund (FWF), DK W1214. B. Moore was supported by the Austrian Science Fund (FWF) under the SFB grant F1303 and the Fonds Québécois de la Recherche sur la Nature et les Technologies (FQRNT).

In this paper, we focus on numerical solvers to find all real solutions on a given box, more specifically, on methods which are based on the Bernstein-Bézier representation of polynomials. There are many reasons for using this representation. The representation is numerically stable [5] by using appropriate algorithms [12], such as de Casteljau algorithm for the computation on subboxes. It has several important properties such as the convex hull property, the variation diminution property and it is affinely invariant [16].

Moreover, various informations about the geometric properties of the graph surface curve can be deduced from its coefficients. Using these properties, many robust methods have been developed for solving system of polynomial equations. One of the first method was the Bézier clipping approach developed by Nishita *et al* [15] which took advantage of the convex hull property. The use of Descartes' rule of sign have lead to many real root isolation algorithms and enclosure algorithms [3, 11, 8]. The *interval projected polyhedron* (IPP) algorithm [19] reformulates the problem of solving a system of polynomial equations into the problem of solving a set of univariate polynomial systems. Various improvements of the IPP method using preconditioning have been described in [13]. These improvement as well as an algorithm for real root isolation based on Bernstein's basis [14] are available in the library SYNAPS. However, as most of the root finding algorithm, this method experience difficulties with roots of higher multiplicities as shown in [10]. Ko and al. also introduce a robust and global method to compute the real roots and multiplicities of a bivariate polynomial system.

The method presented in this paper is an approximation subdivision method to solve a bivariate system of algebraic equations. It extends the approach developed by Bartoň and Jüttler [1, 2]. They compute the best linear approximants of the two polynomials (using L^2 norm) in order to bound these two polynomials between two fat lines. By intersecting these fat lines, they can compute a reduced subbox in which the root lies, and it can be shown that the method provides quadratic convergence for single roots.

In this paper, we use one linear and one quadratic approximant. In this situation, the intersection of the resulting fat line and fat conic can still be computed efficiently, by solving several quadratic equations. In addition, a preprocessing step is applied in order to increase the convergence rate.

2. The algorithm

After stating the root-finding problem, we describe the generic algorithm, which combines clipping and splitting steps.

2.1. The real root-finding problem

We consider two polynomials $f, g \in \mathbb{R}[x, y]$, both given by their Bernstein-Bézier representation with respect to a given axis–aligned box,

$$D_0 = [\xi_1, \xi_2] \times [\eta_1, \eta_2]. \tag{1}$$

Recall that any polynomial $f(x,y)$ of bi-degree (m,n) has a unique representation with respect to the tensor-product Bernstein basis,

$$f(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} b_{ij} U_i^m(x) V_j^n(y), \quad (x,y) \in [\xi_1, \xi_2] \times [\eta_1, \eta_2], \tag{2}$$

where the $b_{ij} \in \mathbb{R}$ are the coefficients and U_i^m and V_j^n are the Bernstein polynomials

$$U_i^m = \binom{m}{i} \frac{(x-\xi_1)^i (\xi_2 - x)^{m-i}}{(\xi_2 - \xi_1)^m} \quad \text{and} \quad V_j^n = \binom{n}{j} \frac{(y-\eta_1)^j (\eta_2 - y)^{n-j}}{(\eta_2 - \eta_1)^n} \tag{3}$$

with respect to the intervals $[\xi_1, \xi_2]$ and $[\eta_1, \eta_2]$, respectively. See e.g. [9, 16] for more details.

We discuss an algorithm for computing all real solutions of the following system of equations

$$\left.\begin{array}{rcl} f(x,y) & = & 0 \\ g(x,y) & = & 0 \end{array}\right\} \tag{4}$$

within the box D_0. In our context, finding a real solution means finding a sufficiently small axis-aligned box (i.e., with a diameter less or equal than a prescribed tolerance ϵ) which contains the root.

2.2. The generic algorithm

The generic algorithm `clip_or_split` (see Algorithm 1) computes all real roots of the algebraic system $f = g = 0$ in the axis-aligned box D_0.

More precisely, the computation of each real root is achieved through the computation of a nested sequence of axis-aligned boxes

$$D_0 \supset D_1 \supset D_2 \supset ... \supset D_n \tag{5}$$

with decreasing diameters, such that $\mathrm{diam}(D_n)$ is smaller than ϵ. The axis-aligned box D_{i+1} is obtained from D_i either by using the Bivariate Quadratic Clipping (`biquadclip`) step (see Algorithm 2) which is discussed in the next section, or by splitting D_i into 4 subboxes ("quadsection").

The splitting step requires the computation of the Bernstein-Bézier representation with respect to the subboxes, which is done with the help of the (numerically stable) algorithm of de Casteljau, see [9, 16]. Though we have not yet implemented this, the algorithm can easily be adapted to interval arithmetic.

The function check_sign(f, D) returns the value 1 if all coefficients b_{ij} of the polynomial f expressed in its Bernstein-Bézier form with respect to D have the same sign (Descartes' rule of signs), and 0 otherwise. The function `split`(D) splits the box D into 4 subboxes.

For any real root, the algorithm returns a box with a diameter less than ϵ with the property that the coefficients of the Bernstein–Bézier representation with respect to this interval do not all have the same sign. Note that this may give "false positive" answers. As an example, Figure 1 shows the initial box and the

Algorithm 1 `clip_or_split`

Require: f, g, D_0, ϵ

$F \Leftarrow \emptyset$ {initialize set of roots}
$L \Leftarrow \{D_0\}$ {initialize list of candidate boxes}
while $L \neq \emptyset$ **do**
  $L' \Leftarrow \emptyset$ {initialize next list of candidate boxes}
  for all $D \in L$ **do**
    $D' \Leftarrow$ `biquadclip`(f, g, D) {apply `biquadclip` to all boxes in L}
    if not $((\mathrm{diam}(D') = 0)$ or $\mathrm{check\_sign}(D') = 1)$ **then**
      if $\mathrm{diam}(D') < \epsilon$ **then**
        $F \Leftarrow F \cup \{D'\}$ {root has been isolated and is added to F}
      else if $\mathrm{diam}(D') \geq \frac{1}{2}\,\mathrm{diam}(D)$ **then**
        $L' \Leftarrow L' \cup$ `split`(D') {box did not shrink enough $\rightarrow$ split}
      else
        $L' \Leftarrow L' \cup \{D'\}$ {box has shrunk enough}
      end if
    end if
  end for
  $L \Leftarrow L'$ {generate new list of candidate boxes}
end while
return F {return set of roots}

first four levels of boxes, which are created by the generic algorithm. Each level corresponds to one execution of the while loop in the algorithm `clip_or_split`.

3. Bivariate quadratic clipping

The bivariate quadratic clipping algorithm, which is summarized in Algorithm 2, consists of three main parts. First, in order to speed up convergence, the system of algebraic equations $f = g = 0$ is converted into another system of algebraic equations $f' = g = 0$. Then, linear and quadratic approximations of the polynomials f' and g as well as bounds of the approximation errors over the box D are computed. Consequently, the curves $f' = 0$ and $g = 0$ can be bounded by linear and quadratic planar curves over the box D. Finally, using these bounds, the intersection of these two planar curves can be bounded and a new reduced axis-aligned box is obtained. A detailed description of these three steps follows.

Step 1: Preprocessing

The aim of the preprocessing step is to increase the convergence rate of the method. This is achieved by replacing the first polynomial f by f', a certain combination of f and g.

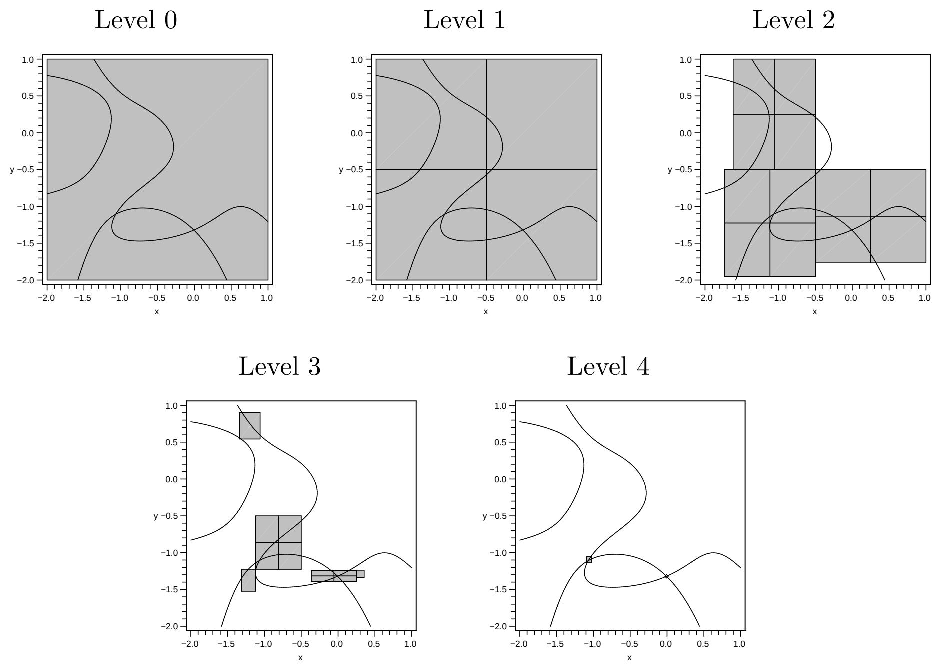


FIGURE 1. The axis-aligned boxes created by algorithm `clip_or_split` for a concrete example.

Let $\varrho = \varrho_0 + \varrho_x x + \varrho_y y$ be a linear polynomial in x, y with the coefficients $\varrho_0, \varrho_x, \varrho_y \in \mathbb{R}$, and

$$f'(x,y) = \varrho(x,y)f(x,y) + (1 - \varrho(x,y))g(x,y). \tag{6}$$

Note that the intersection curve of the two graphs of f and g is always contained in the intersection curve of the two graphs of f' and g.

We choose $\varrho_0, \varrho_x, \varrho_y$ such that the Hessian $\nabla^2 f'$ vanishes in the center

$$(x_c, y_c) = c(D) \tag{7}$$

of the box. This condition leads to the linear system

$$A(x_c, y_c) \begin{pmatrix} \varrho_0 \\ \varrho_x \\ \varrho_y \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

Algorithm 2 `biquadclip`

Require: f, g, D

1. Preprocessing:
   (a) Compute f' from a combination of f and g such that the Hessian vanishes at the centre of the bounding box.
   (b) Keep g.
2. Approximation:
   (a) Compute the best linear approximation l of f' with respect to the L^2 norm on D.
   (b) Compute an error bound ε_l such that
   $$\forall (x,y) \in D: \ |l(x,y) - f'(x,y)| \leq \varepsilon_l$$
   using degree elevation on l. Then the graph f' is bounded by the two planes $l_1 = l - \varepsilon_l$ and $l_2 = l + \varepsilon_l$.
   (c) Compute the best quadratic approximation q of g with respect to the L^2 norm on D.
   (d) Compute an error bound ε_q such that
   $$\forall (x,y) \in D: \ |q(x,y) - g(x,y)| \leq \varepsilon_q$$
   using degree elevation on q. Then the graph g is bounded by the two quadratic surfaces $q_1 = q - \varepsilon_q$ and $q_2 = q + \varepsilon_q$.
3. Clipping:
   (a) Intersect l_1 and l_2 with the plane $z = 0$. A fat line S_1 in the $x - y$ plane where the solutions lie is obtained.
   (b) Intersect q_1 and q_2 with the plane $z = 0$. A fat conic S_2 in the xy plane where the solutions lie is obtained.
   (c) Find the smallest axis-aligned bounding box D' containing all points within the initial bounding box D contained in $S_1 \cap S_2$.
4. return D'

of equations for the unknown coefficients of ϱ, with the matrix

$$A(x_c, y_c) = \left. \begin{pmatrix} \Delta_{xx} & x\Delta_{xx} + 2\Delta_x & y\Delta_{xx} \\ \Delta_{xy} & x\Delta_{xy} + \Delta_y & y\Delta_{xy} + \Delta_x \\ \Delta_{yy} & x\Delta_{yy} & y\Delta_{yy} + 2\Delta_y \end{pmatrix} \right|_{x=x_c, y=y_c} \tag{9}$$

and

$$\Delta_v = \frac{\partial (g-f)}{\partial v}, \quad \Delta_{vw} = \frac{\partial^2 (g-f)}{\partial v \partial w}, \quad v, w \in \{x, y\} \tag{10}$$

If the matrix $A(x_c, y_c)$ given by (9) has full rank, then suitable values for $\varrho_0, \varrho_x, \varrho_y$ can be determined uniquely and therefore f' can be computed. The degree of f' is equal to

$$(\max(m_f, m_g) + 1, \max(n_f, n_g) + 1), \tag{11}$$

where (m_f, n_f) and (m_g, n_g) are the degrees of f and g, respectively.

In order to improve the numerical stability, all computations (including the evaluation of $f' = \varrho f + (1-\varrho)g$) are done in the tensor-product Bernstein basis, requiring product formulas for Bernstein polynomials.

Otherwise if $A(x_c, y_c)$ is singular, i.e.

$$\det A(x_c, y_c) = \left(\Delta_{xx}(\Delta_y)^2 - 2\Delta_{xy}\Delta_x\Delta_y + \Delta_{yy}(\Delta_x)^2\right)\Big|_{x=x_x, y=y_c} = 0 \qquad (12)$$

we choose $f' = f$ and no f' with a vanishing Hessian at the center of the box has been found. In our test implementation, we keep the original function f if the determinant of A is close to zero.

Step 2: Approximation

The best linear approximation l of f' with respect to the L^2 norm can be found using a linear approximation operator as described in [2]. More precisely, the four coefficients of the bilinear Bernstein-Bézier representation of l are found by multiplying the vector of coefficients of f' by a certain constant matrix. In particular, this matrix is independent of the box D.

Next we apply degree elevation to l until we obtain the same degree as for f'. By evaluating the maximum difference between the Bernstein–Bézier coefficients of f' and l, a error bound can be obtained, due to the convex hull property.

The same approach is used to compute the best quadratic approximation q of g and a bound on the approximation error.

Step 3: Clipping

Using the best linear approximation l and the associated error bound ϵ_f, we can bound the graph of f' between the two planes

$$l(x, y) \pm \varepsilon_l = 0. \qquad (13)$$

Similarly, the graph of g is bounded by two quadratic polynomials. When intersecting these bounds with the plane $z = 0$, the zero set of f' is bounded by two lines (fat line) and the zero set of g by two conics (fat conic).

We compute the smallest axis-aligned box $D' \subseteq D$ which contains the intersection of the fat line with the fat conic. This computation can be done efficiently, since it requires solely the solution of linear and quadratic equations. More precisely, the box D' can be found by computing the following critical points within the box D:

1. The corners of the box D,
2. the intersection points between the box boundaries and the lines,
3. the intersection points between the box boundaries and the conics,
4. the intersection points between the lines and the conics, and
5. the x-critical and y-critical points on the conics.

The points of type 4 are always in D'. For the points of type 3 and 5, the points are kept only if they lie between the lines. Similarly, for the point of type 2, we have only to check if they are between the conics. For the points of type 1, we have to check that they are both between the lines and between the conics. By taking the minimum-maximum on x and y of all remaining critical points, we obtain the smallest subbox containing the intersection of the strips.

4. Convergence rate for single roots

We consider the real root-finding problem, as described in Section 2.1, where we assume that the system (4) possesses only the single root $r = (x_0, y_0)$ in the initial box D. In addition we shall assume throughout this section that

$$\det A(x_0, y_0) \neq 0 \tag{14}$$

holds.

For any point $p = (x, y) \in \mathbb{R}^2$ we define f'_p as the result of the preprocessing step, where p is the center of the corresponding bounding box. Then the Hessian of f'_p at p is exactly the null matrix. In addition, we shall assume that the two curves $f'_p = 0$ and $g = 0$ intersect each other transversely at the root r. In other word, $\nabla f'$ and ∇g are linearly independent.

Lemma 1. *There exists a neighbourhood N of the root r such that f'_p exists for all $p \in N$. Moreover, f'_p depends continuously on p.*

Proof. The existence is guaranteed due to (14) and the fact that the matrix $A(x, y)$ depends continuously on x and y. Similarly one can conclude that f'_p depends continuously on p. □

Lemma 2. *For any axis-aligned box D containing the root r and satisfying $D \subseteq N$, where N is the neighborhood of the root r defined in Lemma 1, let l_D be the best linear approximation of f' with respect to the L^2 norm associated with the box D, and let $\varepsilon_{l,D}$ be the corresponding error bound derived in step 2b of algorithm* `biquadclip`*. Then there exists a constant L such that the error bound satisfies*

$$\varepsilon_{l,D} \leq L\,(\mathrm{diam}(D))^3 \tag{15}$$

Proof. We consider the following norms for polynomials of degree defined in (11) which are associated with the box D:

- $\|.\|^D_{BB,\infty}$ is the maximum absolute value of the Bernstein-Bézier coefficients,
- $\|.\|^D_2$ is the L_2–norm associated with D, which is multiplied with the area of the box D, and
- $\|.\|^D_\infty$ is the maximum norm with respect to D.

Let $p = c(D)$ be the centre of the box D. The error bound generated in step 2b of `biquadclip` is equal to

$$\varepsilon_{l,D} = \|f'_p - l_D\|^D_{BB,\infty}. \tag{16}$$

Since all norms on finite dimensional vector spaces are equivalent, there exists constant C_1 and C_2 such that for any polynomial k of the degree defined in (11)

$$\|k\|^D_{BB,\infty} \leq C_1 \|k\|^D_2 \quad \text{and} \quad \|k\|^D_2 \leq C_2 \|k\|^D_\infty \tag{17}$$

Let $T_p^2(f_p')$ be the quadratic Taylor polynomial of f_p' at p. Since the Hessian of f_p' vanishes at p, it is merely a linear polynomial. As l_D is the best linear approximation of f_p' with respect to the norm $\|.\|_2^D$ and using the norm equivalences, we obtain

$$\begin{aligned} \varepsilon_{l,D} &= \|f_p' - l_D\|_{BB,\infty}^D \leq C_1 \|f_p' - l_D\|_2^D \\ &\leq C_1 \|f_p' - T_p^2(f_p')\|_2^D \leq C_1 C_2 \|f_p' - T_p^2(f_p')\|_\infty^D \end{aligned} \tag{18}$$

The maximum norm of $f_p' - T_p^2(f_p')$ can now be bounded by the remainder term of the Taylor expansion,

$$\|f_p' - T_p^2(f_p')\|_\infty^D \leq \frac{1}{6} \underbrace{\max_{\vec{v}\in\mathbb{S}_1, p\in N} \left| \left(\frac{\mathrm{d}^3}{\mathrm{d}\vec{v}^{\,3}} f_p' \right)(p) \right|}_{(*)} (\mathrm{diam}(D))^3, \tag{19}$$

where $\mathrm{d}^3/\mathrm{d}\vec{v}^{\,3}$ denotes the third directional derivative with respect to the direction $\vec{v}$ and $\mathbb{S}_1$ is the unit circle.

Recall that f_p' depends continuously on p and hence also on the box D. If we consider all boxes D contained in N, then the term (*) can be bounded by another constant C_3, which implies the desired result with $L = \frac{1}{6} C_1 C_2 C_3$. □

Lemma 3. *For any axis-aligned box D containing the root r and satisfying $D \subseteq N$, where N is the neighborhood of the root r defined in Lemma 1, let q_D be the best quadratic approximation of g with respect to the L^2 norm associated with the box D, and let $\varepsilon_{q,D}$ be the corresponding error bound derived in step 2.2 of algorithm* `biquadclip`. *Then there exists a constant Q such that the error bound satisfies*

$$\varepsilon_{q,D} \leq Q\ (\mathrm{diam}(D))^3 \tag{20}$$

Proof. Let $p = c(D)$ be the centre of the box D. We consider the same norms as in the previous proof and . Let $T_p^2(g)$ be the quadratic Taylor polynomial of g at p. As q_D is the best quadratic approximation of g with respect to the norm $\|.\|_2^D$ and using the norm equivalences, we obtain

$$\begin{aligned} \varepsilon_{q,D} &= \|g - q_D\|_{BB,\infty}^D \leq C_1 \|g - q_D\|_2^D \\ &\leq C_1 \|g - T_p^2(g)\|_2^D \leq C_1 C_2 \|g - T_p^2(g)\|_\infty^D \end{aligned} \tag{21}$$

The maximum norm of $g - T_p^2(g)$ can now be bounded by the remainder term of the Taylor expansion,

$$\|g - T_p^2(g)\|_\infty^D \leq \frac{1}{6} \underbrace{\max_{\vec{v}\in\mathbb{S}_1, p\in N} \left| \left(\frac{\mathrm{d}^3}{\mathrm{d}\vec{v}^{\,3}} g \right)(p) \right|}_{(*)} (\mathrm{diam}(D))^3. \tag{22}$$

The term (*) can be bounded by another constant C_4, which implies the desired result with $Q = \frac{1}{6} C_1 C_2 C_4$. □

Lemma 4. *In the situation of Lemma* 2 *and* 3*, consider a shrinking sequence of boxes, i.e., diam*$(D) \to 0$*. Then*

$$(\nabla l_D)(r) \to (\nabla f'_r)(r), \quad (\nabla q_D)(r) \to (\nabla g)(r), \quad \textit{and} \quad (\nabla^2 q_D)(r) \to (\nabla^2 g)(r) \quad (23)$$

Proof. This can be proved by a direct computation, similar to Lemma 3.2 in [2]. □

We consider the fat line

$$\mathcal{L}_D = \{(x, y) \in \mathbb{R}^2 : \ |l_D(x, y)| \le \varepsilon_{l,D}\} \quad (24)$$

and the fat conic

$$\mathcal{C}_D = \{(x, y) \in \mathbb{R}^2 : \ |q_D(x, y)| \le \varepsilon_{q,D}\} \quad (25)$$

The quadratic clipping step generates an axis-aligned bounding box D' containing the set

$$\mathcal{L}_D \cap \mathcal{C}_D \cap D. \quad (26)$$

Clearly, if $D \subset N'$, where N' is some neighborhood of the root r, then this set is a subset of

$$\mathcal{Q}_D = \mathcal{L}_D \cap \mathcal{C}_D \cap N'. \quad (27)$$

Corollary 5. *In the situation of the Lemma* 2 *and* 3*, there exists a neighborhood* $N' \subseteq N$ *of the root* r *such that for all boxes* $D \subseteq N'$ *the set* $\mathcal{Q}_D$ *is a curved quadrilateral, which is bounded by two straight line segments which are part of the boundary of* $\mathcal{L}_D$ *and by two conic arcs which are part of the boundary of* $\mathcal{C}_D$*.*

Proof. Both the linear approximation l_D and the quadratic approximation q_D depend continuously on the domain D. As its diameter tends to zero, l_D converges to the linear Taylor polynomial of f'_r at the root r, and q_D converges to the quadratic Taylor polynomial of g at the root r. Since the zero sets of the two Taylor polynomials intersect transversely at the root r, this is also true for the zero sets of l_D and q_D, provided that the domain is small enough. In addition, since the widths of the fat line and the fat conic tend to zero, this property is inherited by the fat line and the fat conic, provided that the D is contained in a certain neighborhood of the root r. □

Remark 6. Recall that the preprocessing keeps the intersection curve of the graph surfaces of f and g. In the limit, the linear approximation l_D represents the osculating plane (in the sense of differential geometry) of this intersection curve at the root r.

Theorem 7. *Consider again the assumptions of Corollary* 5*. Then there exists a constant* C *such that the diameter of the box* D' *generated by the clipping step satisfies*

$$\textit{diam}(D') \le C(\operatorname{diam}(D))^3 \quad (28)$$

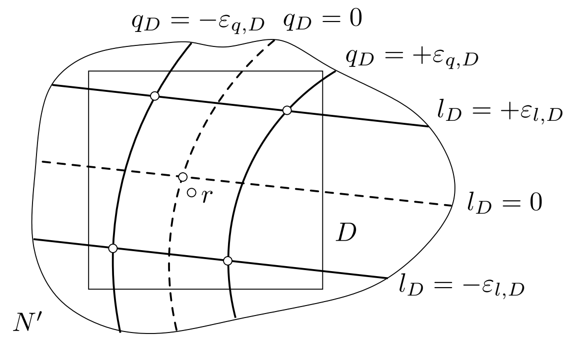


FIGURE 2. Neigboorhood N' of single root with fat line $\mathcal{L}$ and fat conic $\mathcal{C}$.

Proof. The proof is based on the following simple observation which can be proved with the help of the mean value theorem: Consider the restriction of a bivariate function F to an arc of an implicitly defined planar curve G = constant. Then the derivative of the restriction of F to the curve with respect to an arc-length parameterization is

$$\nabla F \cdot \frac{\nabla G^{\perp}}{\|\nabla G\|} \tag{29}$$

where the symbol $^{\perp}$ indicates the rotation of this vector by an angle of $\pi/2$. If the absolute value of this derivative can bounded from below by a positive constant C, then the distance between any two points p, q of the arc defined by $G = 0$ is bounded by $|F(p) - F(q)|/C$.

We apply this observation to the restriction of q_D and of l_D to the curves $l_D =$ constant and $q_D =$ constant. If the box D is contained in a suitable neighborhood N', which is assumed to be sufficiently small, then there exist positive constants C_5 and C_6 not depending on D such that

$$\forall x \in N' : \; |\nabla q_D(x) \cdot \frac{\nabla l_D(x)^{\perp}}{\|\nabla l_D(x)\|}| > C_5 \tag{30}$$

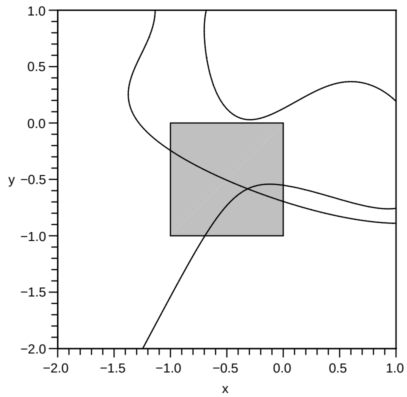
and

$$\forall x \in N' : \; |\nabla l_D(x) \cdot \frac{\nabla q_D(x)^{\perp}}{\|\nabla q_D(x)\|}| > C_6. \tag{31}$$

Indeed the angle between the gradients $\nabla f_r'(r)$ and $\nabla g(r)$ is bounded away from zero, due to the general assumption (see beginning of this section), and this implies these two inequalities for a sufficiently small neighborhood of the root r.

With the help of these two inequalities we conclude that no point of the boundary of the quadrilateral described in Corollary 5 has a distance greater that

$$\frac{\varepsilon_{q,D}}{C_5} + \frac{\varepsilon_{l,D}}{C_6} \tag{32}$$



Level	diam(D_i)	
	Without prep.	With prep.
1	1.23	1.34
2	0.23	0.16
3	0.14×10^{-01}	0.13×10^{-02}
4	0.48×10^{-04}	0.77×10^{-09}
5	0.54×10^{-09}	

TABLE 1. Example 1 – Diameters of bounding boxes for a single root

from the intersection point of $l_D = 0$ and $q_D = 0$. For the points on the straight boundary, this bound is obtained by first bounding the distance to the intersection of the boundary with the curve $q_D = 0$ and then bounding the distance to the intersection point, and similarly for the curved boundary. Consequently, using Lemma 2 and 3, the curved quadrilateral is contained in an axis-aligned bounding box with diameter

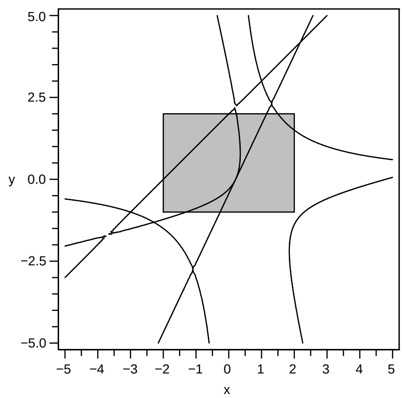
$$\sqrt{2}(\frac{Q}{C_5} + \frac{L}{C_6})(\mathrm{diam}(D))^3. \tag{33}$$

This proves the theorem since the intersection with D does not lead to an increased diameter of the bounding box. □

5. Experimental results

Example 1. (Single root) We consider a bivariate polynomials f of bidegree (3,3) and q of bidegree (5,5) on the box $D = [-1, 0] \times [-1, 0]$ as shown in Table 5. This algebraic system has only one single root in D. Table 5 also gives the diameter of the reduced box obtained using the BQC algorithm after every step. With the preprocessing step, we obtain convergence rate of 3, otherwise we obtain a convergence rate of 2.

Example 2. (Double roots) In this example, we investigate the convergence rate of the algorithm for double roots. We consider f and g with bidegree (2,2) and (3,3) respectively, on the box $[-2, 2] \times [-2, 2]$ as shown in Table 2. In the same table, the diagonal's lengths of the boxes at every step of the algorithm are shown. The results have been obtained in Maple in order to demonstrate that the convergence rate is 1.5 for double roots.



Level	diam(D_i)	
	Without prep.	With prep.
1	0.38	0.35
2	0.82×10^{-02}	0.50×10^{-02}
3	0.13×10^{-03}	0.83×10^{-05}
4	0.22×10^{-05}	0.56×10^{-09}
5	0.37×10^{-07}	0.31×10^{-15}
6	0.61×10^{-09}	0.37×10^{-24}
7	0.10×10^{-10}	
8	0.16×10^{-12}	
9	0.27×10^{-14}	
10	0.44×10^{-16}	
11	0.74×10^{-18}	
12	0.12×10^{-19}	
13	0.20×10^{-21}	
14	0.33×10^{-23}	

TABLE 2. Example 2 – Diameters of bounding boxes for a double root

Example 3. (Transition from single roots to double roots) We use the example from [2] (example 4.3) to study the behavior of the algorithm when approaching a double root. Consider the following sequence of polynomial systems (k=0,1,2,...):

$$\begin{array}{rcl} 0 & = & x^2 + y^2 - \left(1 + \frac{1}{10^k}\right) \\ 0 & = & xy - \frac{1}{2} \end{array} \tag{34}$$

over the box $D = [0,2] \times [0,2]$. For small k, the system of polynomials equations has two single roots in D. Increasing the value of k makes the two single roots converge to a double root. For a required accuracy of 10^{-8} and different values of k, the diameters of the largest box at each level of the bivariate clipping algorithm, with and without preprocessing, are shown in Table 3. We observe that the preprocessing step has a significant impact on the convergence rate when approaching a double root. For instance, if $k = 6$, then 7 steps are sufficient with preprocessing, while 13 steps are needed without.

6. Concluding remarks

A new method for the computation of all real roots of a system of two bivariate polynomials in a given box have been presented. It was shown that the convergence rate for single roots is 3 if a suitable preprocessing is used. Moreover, experimental results indicate that the convergence rate is superlinear for double roots.

Planned future work includes the use of interval arithmetic or similar techniques to make the approach more robust. In addition, we are planning to investigate suitable criteria to turn on and off the preprocessing step, in order to reduce computational time. Indeed, for well-separated single roots, the proposed preprocessing is too costly and should not be used, since the "normal" method already

	k=0		k=3		k=6	
Level	No prep.	With prep.	No prep.	With prep.	No prep.	With prep.
1	2.4	2.4	2.0	1.9	2.0	1.9
2	0.4	0.2	0.4	0.2	0.4	0.2
3	0.5×10^{-1}	0.4×10^{-2}	0.2	0.6×10^{-1}	0.2	0.3×10^{-1}
4	0.5×10^{-3}	0.1×10^{-5}	0.9×10^{-1}	0.5×10^{-1}	0.8×10^{-1}	0.4×10^{-2}
5	0.9×10^{-7}	0.7×10^{-13}	0.4×10^{-1}	0.7×10^{-4}	0.4×10^{-1}	0.1×10^{-2}
6	0.2×10^{-14}		0.1×10^{-1}	0.6×10^{-10}	0.2×10^{-1}	0.6×10^{-7}
7			0.2×10^{-2}		0.7×10^{-2}	0.1×10^{-15}
8			0.3×10^{-4}		0.3×10^{-2}	
9			0.7×10^{-8}		0.1×10^{-2}	
10					0.6×10^{-3}	
11					0.9×10^{-4}	
12					0.2×10^{-5}	
13					0.1×10^{-8}	

TABLE 3. Example 3 – Diameters of bounding boxes for two neigbouring single roots with and without preprocessing.

converges very fast. The preprocessing step should be applied for double roots or situations close to it.

Finally, we would like to extend the approach to higher dimensions. For more than two polynomials, the intersections of the graph surfaces of the polynomials defines a curve in a higher-dimensional space. Instead of the osculating plane (cf. Remark 6) one should now use the osculating 2-plane, which can be represented as the intersection of $n-1$ planes, where n is the number of polynomials. Hence, in order to obtain a suitable quadratic splitting step, it should sufficient to use $n-1$ linear and 1 quadratic approximations.

References

[1] M. Bartoň and B. Jüttler. Computing roots of polynomials by quadratic clipping. *Computer Aided Geometric Design*, 24(3):125–141, April 2007.

[2] M. Bartoň and B. Jüttler. Computing roots of systems of polynomials by linear clipping. Technical Report 2007-18, SFB F013 Technical Report, August 2007. Available at *http://www.sfb013.uni-linz.ac.at*.

[3] G. Elber and M.-S. Kim. Geometric constraint solver using multivariate rational spline functions. In *The Sixth ACM/IEEE Symposium on Solid Modeling and Applications*, pages 1–10, Ann Arbor, MI, 2001.

[4] M. Elkadi and B. Mourrain. Symbolic-numeric methods for solving polynomial equations and applications. In A. Dickenstein and I. Z. Emiris, editors, *Solving Polynomial Equations*, volume 14 of *Algorithms and Computation in Mathematics*, pages 125–168. Springer, 2004.

[5] R. T. Farouki and T. N. T. Goodman. On the optimal stability of the Bernstein basis. *Math. Comp.*, 65:1553–1566, 1996.

[6] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases (F_4). *J. Pure Appl. Algebra*, 139(1-3):61–88, 1999. Effective methods in algebraic geometry (Saint-Malo, 1998).

[7] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5). In *Proceedings of the 2002 International Symposium on Symbolic and Algebraic Computation*, pages 75–83 (electronic), New York, 2002. ACM.

[8] J. Garloff and A. P. Smith. Investigation of a subdivision based algorithm for solving systems of polynomial equations. *Nonlinear Anal.*, 47(1):167–178, 2001.

[9] J. Hoschek and D. Lasser. *Fundamentals of Computer Aided Geometric Design*. AK Peters, Wellesley, MA, 1993.

[10] K. H. Ko, T. Sakkalis, and N. M. Patrikalakis. Nonlinear polynomial systems: multiple roots and their multiplicities. In *Proc. Shape Modeling International (SMI)*, 2004.

[11] D. Lutterkort and J. Peters. Optimized refinable enclosures of multivariate polynomial pieces. *Computer Aided Geometric Design*, 18:851–863, 2001.

[12] E. Mainar and J. M. Peña. Evaluation algorithms for multivariate polynomials in Bernstein-Bézier form. *Journal Approximation Theory*, 143:44–61, 2006.

[13] B. Mourrain and J.P. Pavone. Subdivision methods for solving polynomial equations. *Journal of Symbolic Computation*, 2008. accepted manuscript.

[14] B. Mourrain, F. Rouillier, and M.-F. Roy. The Bernstein basis and real root isolation. In *Combinatorial and computational geometry*, volume 52 of *Math. Sci. Res. Inst. Publ.*, pages 459–478. Cambridge Univ. Press, Cambridge, 2005.

[15] T. Nishita, T. Sederberg, and M. Kakimoto. Ray tracing trimmed rational surface patches. *Computer Graphics*, 24(4):337–345, 1990.

[16] H. Prautzsch, W. Boehm, and M. Paluszny. *Bézier and B-Spline techniques*. Springer, Berlin, 2002.

[17] F. Rouillier. Solving zero-dimensional systems through the rational univariate representation. *Appl. Algebra Engrg. Comm. Comput.*, 9(5):433–461, 1999.

[18] F. Rouillier and P. Zimmermann. Efficient isolation of polynomial's real roots. *J. Comput. Appl. Math.*, 162(1):33–50, 2004.

[19] E.C. Sherbrooke and Nicholas M. Patrikalakis. Computation of the solutions of nonlinear polynomial systems. *Computer Aided Geometric Design*, 10(5):379–405, October 1993.

Bert Jüttler
Johannes Kepler University, Institute of Applied Geometry
Altenberger Str. 69, 4040 Linz, Austria
e-mail: bert.juettler@jku.at

Brian Moore
Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Science, Altenberger Str. 69, A-4040, Linz, Austria
e-mail: brian.moore@ricam.oeaw.ac.at