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Parameterization of DIAMOND Surfaces*

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Abstract

We discuss rational parameterizations of surfaces from the special class of DIAMOND surfaces. They are characterized by the property that the support function is a rational function of the coordinates and of a given non-degenerate quadratic form. Geometrically, they can be analyzed with the help of the associated dual surface. That surface is a projection of the intersection of three hypersurfaces in five-dimensional space into three-dimensional space. The three hypersurfaces are two quadratic cones (with two-dimensional generators) and an axial monoid. The class of DIAMOND surfaces includes the offset surfaces of non-developable quadric surfaces.

We show how to construct rational parameterizations of these surfaces. If the quadratic form is diagonalized and has rational coefficients, then these parameterizations are almost always described by rational functions with rational coefficients.

Key words: Parameterization of surfaces, support function, offset surface

1. Introduction

The support function representation of a surface is one of the classical tools in the field of convex geometry (see Bonnesen and Fenchel, 1987; Groemer, 1996; Gruber and Wills, 1993). It describes the surface as the envelope of its tangent planes, where the distance between the tangent plane and the origin is specified by a function of the unit normal

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vector. This representation is particularly well suited for discussing offsets surfaces, since the offsetting operation corresponds simply to the addition of constants. See Gravesen et al. (2008a,b) for a detailed discussion of surfaces with polynomial and rational support functions.

The analysis and parameterization of offset curves and surfaces via techniques from symbolic computation and algebraic geometry has been the central topic subject of several publications. Alcazar et al. (2007) apply a method for computing critical sets of algebraic surfaces to the offsetting problem. Alcazar and Sendra (2007) study the local shape of offsets to algebraic curves. Landsmann et al. (2001) present an algorithm for parameterizing canal surfaces by decomposing a polynomial into a sum of squares. Canal surfaces can be seen as (generalized) offsets of space curves. Arrondo et al. (1997) give a theoretical analysis of the rationality and unirationality of offsets to hypersurfaces. Of course it is also possible to apply results about the parameterization of general rational surfaces (Schicho, 1998) to the case of offset surfaces.

Due to their importance in applications in Computer Aided Design, the case of offsets to quadric surfaces has attracted special attention. Moreover, these surfaces are (after planes) the simplest instance of offsets to a class of algebraic surfaces. With the help of techniques from Laguerre Geometry, Peternell and Pottmann (1998) derive a rational parameterization of the offsets of quadric surfaces. First, the quadric surface and its offsets are represented as the envelope of a one-parameter-family of quadratic cones of revolution. Then a parameterization of the envelope is found by a geometric algorithm, which involves the decomposition of a polynomial into a sum of squares, similar to the case of canal surfaces, see Landsmann et al. (2001). This decomposition requires a suitable field extension, which may make the use of exact symbolic computation techniques more difficult. Sendra and Sendra (2000) discuss generalized offsets of irreducible quadrics and show how to obtain rational parameterizations (if available) from the parameterization of the original quadric.

In this paper we apply the support function representation for surfaces to the specific class of DIAMOND surfaces, which includes the offsets of quadric surfaces. We show how to generate rational parameterizations of these surfaces. Except for certain special cases (the offsets to two-sheeted hyperboloids of revolution), the presented algorithm does not require field extensions. Consequently, it produces parameterizations with rational coefficients, provided that the input surfaces have also been specified by support functions with rational coefficients.

The remainder of this paper is structured in five parts. The next section recalls the concept of the dual representation of non-developable algebraic hypersurfaces and analyzes its relation to support functions. Section 3 introduces the class of DIAMOND surfaces, and Section 4 discusses its parameterization via the envelope operator. The fifth section presents an algorithm for parameterizing the intersection of two special hyperquadrics in four-dimensional space and applies it to the parameterization problem of DIAMOND surfaces. Finally we conclude the paper.

2. Dual representation of algebraic surfaces and support functions

We consider algebraic surfaces in the three–dimensional Euclidean space, which is identified with $\mathbb{R}^3$. Sometimes it will be helpful to use the projective closure $\bar{\mathbb{R}}^3$ of this space, which is obtained by adding a point at infinity for each equivalence class of parallel lines, a line at infinity for each equivalence class of parallel planes, and a plane at infinity collecting these points and lines.

Recall that a non-developable algebraic surface $\mathcal{S}$ in $\mathbb{R}^3$ has a *dual representation* of the form

$$F(h, \mathbf{n}) = 0 \tag{1}$$

where F is a homogeneous polynomial in h and $\mathbf{n} = (n_1, n_2, n_3)^\top$. The degree of F is called the *class* of the surface $\mathcal{S}$. The set of all planes

$$T_{h,\mathbf{n}} = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{n}^\top \mathbf{x} = h\}, \quad F(h, \mathbf{n}) = 0, \tag{2}$$

forms the system of the *tangent planes* of the surface $\mathcal{S}$. The vector $\mathbf{n}$ is the normal vector. If $\mathbf{n}^\top \mathbf{n} = 1$, then the value of h is the oriented distance of the tangent plane to the origin.

If the partial derivative $\partial F / \partial h$ does not vanish at $(h_0, \mathbf{n}_0) \in \mathbb{R}^4$ and $F(h_0, \mathbf{n}_0) = 0$ holds, then (1) implicitly defines a function

$$\mathbf{n} \mapsto h(\mathbf{n}), \tag{3}$$

which is well-defined in a certain neighborhood of $(h_0, \mathbf{n}_0) \in \mathbb{R}^4$. The restriction of this function to the unit sphere

$$\mathbb{S} = \{\mathbf{n} \in \mathbb{R}^3 : \ \mathbf{n}^\top \mathbf{n} = 1\} \tag{4}$$

is then called the *support function* of the surface $\mathcal{S}$.

Alternatively, we may consider

$$h : n_1 : n_2 : n_3 = 1 : x_1 : x_2 : x_3 \tag{5}$$

as homogeneous coordinates in $\mathbb{R}^3$. Then Eq. (1) defines the *dual surface* $\mathcal{D}$ associated with $\mathcal{S}$. The dual surface has the equation

$$F(1, \mathbf{x}) = 0. \tag{6}$$

The points (resp. tangent planes) of this surface $\mathcal{D}$ are obtained by applying the polarity with respect to the imaginary unit sphere to the tangent planes (resp. points) of the surface $\mathcal{S}$. This polarity identifies the homogeneous coordinates of points and of planes according to (5).

If a *parametric representation* of a surface is known, then the support function can be obtained as shown in the following example.

Example 1. We consider the algebraic surface of order 4 which possesses the quadratic parameterization $\mathbf{p}(u, v) = (u + v, u^2, v^2)$. The dual representation

$$F(h, \mathbf{n}) = n_1^2(n_2 + n_3) + 4hn_2n_3 = 0 \tag{7}$$

can be found by eliminating u and v from the three equations

$$\mathbf{n}^\top \frac{\partial}{\partial u}\mathbf{p} = 0, \quad \mathbf{n}^\top \frac{\partial}{\partial v}\mathbf{p} = 0, \quad \mathbf{n}^\top \mathbf{p} - h = 0. \tag{8}$$

Consequently, as F is a cubic homogeneous polynomial, this surface has the class three. The dual surface $\mathcal{D}$ is a cubic monoid (see Johansen et al., 2008) with a unique singular point at the origin. The support function of the surface is the function

$$h(\mathbf{n}) = -\frac{n_1^2(n_2 + n_3)}{4n_2n_3}. \tag{9}$$

In this case, we have obtained even a unique rational support function. This was possible, as the given parameterization describes a non–developable quadratic polynomial surface (see Gravesen et al., 2008a).

On the other hand, the support function can also be obtained directly from the *implicit equation* of a surface, as shown in the next example.

Example 2. The quadric surface with the equation

$$f(\mathbf{x}) = x_1^2 + x_2^2 + \frac{1}{2}x_3^2 - 1 = 0 \tag{10}$$

is an ellipsoid of revolution with the principal radii 1, 1 and $\frac{1}{\sqrt{2}}$. The dual representation

$$F(h, \mathbf{n}) = n_1^2 + n_2^2 + 2n_3^2 - h^2 \tag{11}$$

can be found by eliminating the four variables λ and $\mathbf{x} = (x_1, x_2, x_3)$ from the five $(= 3 + 1 + 1)$ equations

$$\mathbf{n} - \lambda \nabla f = 0, \quad \mathbf{n}^\top \mathbf{x} - h = 0, \quad f(\mathbf{x}) = 0. \tag{12}$$

The support functions of the surface takes the form

$$h(\mathbf{n}) = \pm\sqrt{n_1^2 + n_2^2 + 2n_3^2}. \tag{13}$$

Finally we note that certain geometric operations correspond to simple modifications of the support functions:

(1) *Rotations* can be *composed* with the support function; the support function of $\varrho(\mathcal{S})$ is $h \circ \varrho$, where ϱ is a rotation around the origin.
(2) A *translation* by a vector $\vec{\mathbf{v}}$ correspond to the *addition* of the homogeneous *linear polynomial* $\vec{\mathbf{v}}^\top \mathbf{n}$ to the support function.
(3) The one-sided *offset* of a surface at distance δ can be obtained by adding the *constant* δ to the support function.
(4) The *reciprocal* support function $(1/h)$ describes the surface which is obtained by applying the polarity with respect to the unit sphere to the *pedal surface*.

In the remainder of the paper we consider a class of surfaces with a specific form of the support function.

3. DIAMOND surfaces: A special class of support functions

We consider support functions of the form

$$h(\mathbf{n}) = R(Q, \mathbf{n}) \tag{14}$$

where $Q = \sqrt{\mathbf{n}^\top \mathbf{D} \mathbf{n}}$, $\mathbf{D} = \text{diag(1,b,c)}$ with $b, c \neq 0$, and R is a rational function of its four arguments Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$. We can rewrite this function as

$$h(\mathbf{n}) = \frac{p_1(Q, \mathbf{n}) + p_2(Q, \mathbf{n})}{q(Q, \mathbf{n})}, \tag{15}$$

where the two functions p_2 and q are homogeneous polynomials in Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$ of the even degree $2d$, and p_1 is a homogeneous polynomial of the odd degree $2d + 1$, where d is a nonnegative integer. This can be proved by exploiting the observation that the terms can be multiplied by multiples of $\mathbf{n}^\top \mathbf{n}$, since this expression equals 1 on the unit sphere $\mathbb{S}$, similar to the proof of Lemma 2 in Gravesen et al. (2008a).

Clearly, the class of surfaces with support functions of the form (14) includes non-developable quadric surfaces and their offsets, see Example 2.

Remark 3. More generally, one might consider square roots of a general quadratic form. In order to simplify the notation, we assume that it has been diagonalized and scaled such that the first coefficient is equal to 1.

The corresponding dual equation (1) can be found by eliminating N and Q from the three equations

$$p_1(Q,\mathbf{n}) + p_2(Q,\mathbf{n})N - h\,q(Q,\mathbf{n}) = 0, \quad N^2 - \mathbf{n}^\top\mathbf{n} = 0, \quad Q^2 - \mathbf{n}^\top\mathbf{D}\mathbf{n} = 0. \qquad (16)$$

The left-hand sides of all equations are homogeneous polynomials in h, N, Q and $\mathbf{n} = (n_1, n_2, n_3)^\top$. Consequently, the elimination of N and Q produces a homogeneous polynomial F. Note that the dual equation (1) then corresponds to the four support functions

$$h_{\epsilon_1,\epsilon_2}(\mathbf{n}) = \frac{p_1(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n}) + \epsilon_2 p_2(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n})}{q(\epsilon_1\sqrt{\mathbf{n}^\top\mathbf{D}\mathbf{n}},\mathbf{n})}, \quad \epsilon_1,\epsilon_2 \in \{\pm 1\}, \qquad (17)$$

due to the sign ambiguities in N and D. This gives two pairs of support functions describing the same surface. Indeed, the support functions $h(\mathbf{n})$ and $h^*(\mathbf{n}) = -h(-\mathbf{n})$ describe the same surface, but with opposite orientations of the normals.

Finally we analyze the dual surface $\mathcal{D}$ associated with this support function. The three equations (16) define three hypersurfaces in the five dimensional space with the homogeneous coordinates $h : N : Q : n_1 : n_2 : n_3$.

The first equation describes a hypersurface of degree $2d + 1$, where each point of the line

$$n_1 = n_2 = n_3 = Q = 0 \qquad (18)$$

has multiplicity $2d$. It is therefore a very special instance of a monoid hypersurface, see Johansen et al. (2008). We call this surface an *axial monoid* with axis (18) .

The remaining two surfaces describe two quadratic hypercones with two–dimensional generators and one-dimensional singular loci.

The first three unit points of the projective coordinate system span three lines. One of them is the axis of the axial monoid, while the other two lines are the singular loci of the hypercones.

The three hypersurfaces intersect in a two dimensional surface. The dual surface is obtained as its image by a central projection with the center line spanned by the two points $(0 : 1 : 0 : 0 : 0 : 0)$ and $(0 : 0 : 1 : 0 : 0 : 0)$ into the 3-plane $N = R = 0$.

Summing up, this motivates us to introduce the following notion.

Definition 4. Any surface $\mathcal{S}$ defined by the support function (14), which involves rational expressions and a square root of a single quadratic form, will be called a **DIAMOND surface**. It is the Dual surface of Intersections of Axial MONoids with special quaDratic cones, centrally projected into a 3-space.

Example 5. We consider the support function

$$h(\mathbf{n}) = n_1\sqrt{n_1^2 + n_2^2 + 2n_3^2}. \qquad (19)$$

In this case, we have $d = 1$, $\mathbf{D} = \mathrm{diag}(1, 1, 2)$ and

$$p_1(Q,\mathbf{n}) = 0, \quad p_2(Q,\mathbf{n}) = n_1 Q, \quad q(Q,\mathbf{n}) = n_1^2 + n_2^2 + n_3^2. \qquad (20)$$

After eliminating Q and N from the equations (16) we arrive at the dual representation of the surface S,

$$F(h,\mathbf{n}) = (n_1^2 + n_2^2 + n_3^2)h^2 - n_1^2(n_1^2 + n_2^2 + 2n_3^2). \qquad (21)$$

4. Parameterization of DIAMOND surfaces

First we introduce an operator that assigns to each support function a parameterization of the corresponding surface, where the parameter domain is the sphere or a subset thereof, cf. Gravesen et al. (2008a,b).

Definition 6. Let $U \subset \mathbb{S}$ be an open subset of the unit sphere[1] and $h \in C^\infty(U, \mathbb{R})$ be a support function. We define the **envelope operator**

$$\mathcal{E}: \; C^\infty(U, \mathbb{R}) \to C^\infty(U, \mathbb{R}^3) \tag{22}$$

which is defined via

$$\mathcal{E}(h): \; \mathbf{n} \mapsto h(\mathbf{n})\mathbf{n} + (\nabla_S h)(\mathbf{n}) \tag{23}$$

with the intrinsic gradient

$$(\nabla_S h)(\mathbf{n}) = (\nabla h)(\mathbf{n}) - \left(\mathbf{n}^\top [(\nabla h)(\mathbf{n})]\right) \mathbf{n}. \tag{24}$$

Remark 7. The intrinsic gradient (24) is the projection of the gradient in $\mathbb{R}^3$ onto the unit sphere, where we assume that h has been extended to the embedding space. Eq. (23) gives the envelope of the two-parameter family of planes $T_{h(\mathbf{n}),\mathbf{n}}$, see (2).

For any parameterization $\nu : \Omega \to U$ of $U \subseteq \mathbb{S}$ with the domain $\Omega \subseteq \mathbb{R}^2$, the mapping $\nu \circ \mathcal{E}(h) : \Omega \to \mathbb{R}^3$ is a parameterization of the corresponding open subset of the surface $\mathcal{S}$ in three–dimensional space. Clearly, if we apply the envelope operator $\mathcal{E}$ to a rational support function h (see Gravesen et al., 2008a) and compose the result with a rational parameterization ν of the sphere, then we obtain a rational parameterization $\nu \circ \mathcal{E}(h)$ of the corresponding surface $\mathcal{S}$.

In the case of DIAMOND surfaces, which have special non-rational support functions, we have the following result.

Lemma 8. *If the five bivariate polynomials $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}[u, v]$ satisfy the two identities*

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 \quad \textit{and} \quad x_1^2 + b\, x_2^2 + c\, x_3^2 = x_5^2 \tag{25}$$

then the mapping

$$(u, v) \mapsto \mathcal{E}(h) \left(\frac{x_1(u,v)}{x_4(u,v)}, \frac{x_2(u,v)}{x_4(u,v)}, \frac{x_3(u,v)}{x_4(u,v)} \right) \tag{26}$$

is a piecewise rational parameterization of the DIAMOND surface which is defined by the support function $h(\mathbf{n})$.

Proof. If the support function has the form (14), then $\mathcal{E}(h)$ as defined in (23) contains only rational functions of $\mathbf{n}$ and $\sqrt{\mathbf{n}^\top \mathbf{D} \mathbf{n}}$. Consequently, if we have five polynomials satisfying the two equations, then the rational surface $(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4})$ parameterizes the unit sphere, and the square root $\sqrt{\mathbf{n}^\top \mathbf{D} \mathbf{n}}$ in (26) can be replaced by $|x_5|$. $\square$

The next section discusses how to generate quintuples of bivariate polynomials that satisfy the assumptions of the Lemma.

[1] U is the intersection of an open set in $\mathbb{R}^3$ with the unit sphere $\mathbb{S}$.

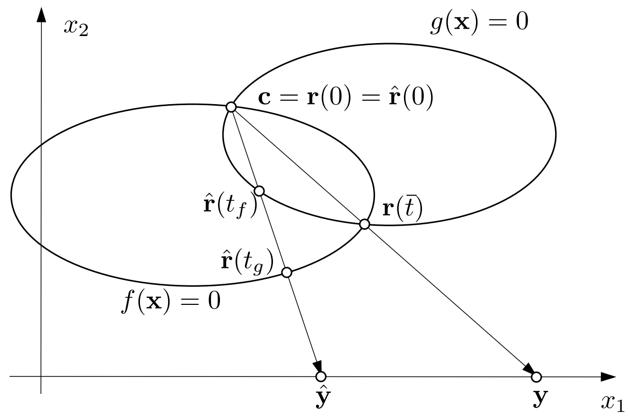


Figure 1. Stereographic projection of two intersecting quadrics

5. Intersections of special hyperquadrics in four-dimensional space

The two identities (25) define two quadric surfaces

$$\begin{aligned} f(x_1,x_2,x_3,x_4,x_5) &= x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0 \\ g(x_1,x_2,x_3,x_4,x_5) &= x_1^2 + b\,x_2^2 + c\,x_3^2 - x_5^2 = 0 \end{aligned} \tag{27}$$

in four–dimensional real projective space with homogeneous coordinates $x_1 : x_2 : x_3 : x_4 : x_5$. The intersection is a two-dimensional Del Pezzo surface (see Schicho, 2005).

We assume that the input satisfies $b \neq c$. See Remark 13 for a discussion of the special case $b = c$.

We parameterize the intersection by applying the following algorithm.

Algorithm 9. Parameterization of the intersection of (27), where $b \neq c$.

(1) Find a point $\mathbf{c}$ on the intersection of the two quadrics.
(2) Apply stereographic projection with center $\mathbf{c}$ into a three-dimensional subspace to the intersection surface. This gives a cubic surface k.
(3) Find a straight line $\mathbf{l}$ on the cubic surface k and parameterize it linearly with parameter u.
(4) For each point $\mathbf{l}(u)$ on the line, compute the tangent plane $\mathbf{q}(u)$ of the cubic k.
(5) The intersection of the tangent plane $\mathbf{q}(u)$ with the cubic surface k gives a conic section, which is parameterized with the parameter v.
(6) Lift the parameterization of k back into the five-dimensional space.

Now we describe the six steps of the algorithm in more detail.

Step (1) We simply observe that the point $\mathbf{c} = (1, 0, 0, 1, 1)^\top$ lies on both quadrics, hence it is also contained in the intersection.

Step (2) We apply stereographic projection with center $\mathbf{c}$ and project the intersection of both quadrics into the plane $x_5 = 0$. More precisely, for each point $\mathbf{y} = (y_1, y_2, y_3, y_4, 0)$ we consider the line

$$\mathbf{r}(t) = (1-t)\mathbf{c} + t\mathbf{y}, \tag{28}$$

see Figure 1 for a two-dimensional sketch. A point $\mathbf{y}$ belongs to the image of the intersection if and only if the line (28) intersects both quadrics in the same point. Equivalently, there exists a parameter $\bar{t} \neq 0$ such that the equations

$$f((1-\bar{t})\mathbf{c} + \bar{t}\mathbf{y}) = 0, \quad g((1-\bar{t})\mathbf{c} + \bar{t}\mathbf{y}) = 0 \tag{29}$$

are simultaneously satisfied. This can characterized by the resultant

$$k(\mathbf{y}) = \mathrm{Res}(\frac{1}{t}f((1-t)\mathbf{c} + t\mathbf{y}), \frac{1}{t}g((1-t)\mathbf{c} + t\mathbf{y}), t), \tag{30}$$

where we factored out the root $t = 0$ which corresponds to the trivial intersection $\mathbf{c}$. By evaluating the resultant we obtain the equation

$$k(\mathbf{y}) = cy_3^2 y_4 + by_2^2 y_4 - y_1 y_4^2 + y_1^2 y_4 + (1-b)y_1 y_2^2 + (1-c)y_1 y_3^2 \tag{31}$$

which defines a cubic surface in three-dimensional real projective space with homogeneous coordinates $y_1 : y_2 : y_3 : y_4$.

Step (3) A close inspection reveals the fact that the cubic surface $k(\mathbf{y})$ contains the straight line $\mathbf{l}(u) = (0, 1, u, 0)^\top$. Indeed, this line is the intersection of the two-dimensional tangent plane at the center $\mathbf{c}$ of the surface in five-dimensional space with the image hyperplane.

Step (4) Now we move the tangent plane of k along this line and intersect it with the cubic. This technique is closely related to one of the local parameterization techniques for cubic surfaces that were described by Szilágyi et al. (2006). In this special case the computations become much simpler, as a line on the cubic surface is known. For any value of u, the tangent plane can be parameterized by

$$\mathbf{q}(u, s_1, s_2) = \mathbf{l}(u) + s_1\mathbf{v}_1(u) + s_2\mathbf{v}_2(u) \tag{32}$$

where $\mathbf{v}_1 = (0, 0, 1, 0)^\top$ and $\mathbf{v}_2 = (cu^2 - b, 0, 0, (1-c)u^2 + 1 - b)^\top$.

Step (5) The intersection of $\mathbf{q}(u, s_1, s_2)$ with $k(\mathbf{y})$ gives the equation

$$k(\mathbf{q}(u, s_1, s_2)) = 2u(b-c)s_1 + (b-c)s_1^2 + (u^2+1)(b+cu^2)(b-1+u^2(c-1))s_2^2, \tag{33}$$

which defines a conic section in the s_1, s_2-plane. The conic-section is non-degenerate, as $b \neq c$ was assumed. We parameterize each of these conic sections by intersecting it with lines through $(s_1, s_2) = (0, 0)$, which gives

$$\begin{gathered} s_1 = \frac{1}{N}2u(c-b), \quad s_2 = \frac{1}{N}2uv(c-b), \quad \text{where} \\ N = b-c+v^2(c^2u^4-2u^2b-cu^2+b^2u^2+c^2u^6-bu^4-cu^6-2cu^4+2bcu^2+2bcu^4-b+b^2) \end{gathered} \tag{34}$$

Finally we obtain the parameterization

$$\begin{aligned} y_1 &= 2uv(b-c)(cu^2+b) \\ y_2 &= 2bcv^2u^4 + 2bcv^2u^2 + c^2v^2u^4 - cv^2u^6 + c^2v^2u^6 - bv^2 - bv^2u^4 \\ &\quad + b^2v^2u^2 - 2bv^2u^2 - 2cv^2u^4 - cv^2u^2 + b^2v^2 - c + b \\ y_3 &= u(2bcv^2u^4 + 2bcv^2u^2 + c^2v^2u^4 - v^2cu^6 + c^2v^2u^6 - bv^2 - bv^2u^4 \\ &\quad + b^2v^2u^2 - 2bv^2u^2 - 2cv^2u^4 - cv^2u^2 + b^2v^2 + c - b) \\ y_4 &= 2uv(b-c)(-1+b+cu^2-u^2) \end{aligned}$$

of the cubic surface.

Step (6) We lift the parameterization back into the five-dimensional space. We substitute the parameterization $\mathbf{y}(u,v) = (y_1(u,v), y_2(u,v), y_3(u,v), y_4(u,v), 0)$ into (29) and solve this equation for $\bar{t}(u,v)$. Finally, the parameterization of the intersection is given by

$$\mathbf{p}(u,v) = (1-\bar{t}(u,v))\mathbf{c} + \bar{t}(u,v)\mathbf{y}(u,v). \tag{35}$$

Example 10. As in Example 5 we consider again the surface defined by the support function $h(\mathbf{x}) = x_1\sqrt{x_1^2 + x_2^2 + 2x_3^2}$, see Fig. 2a. This support function fulfills the requirements of the parameterization algorithm with $b = 1$ and $c = 2$. After the first stereographic projection we obtain the cubic surface

$$k(\mathbf{y}) = -y_1y_3^2 - y_1y_4^2 + y_4y_1^2 + y_4y_2^2 + 2y_4y_3^2. \tag{36}$$

Following the next step of the algorithm, we compute the tangent planes along the line $\mathbf{l}(u) = (0,1,u,0)^\top$ and intersect them with the cubic. After parameterizing them we obtain a parameterization of the cubic surface,

$$\begin{aligned} y_1 &= -2uv(1+2u^2) \\ y_2 &= 2v^2u^6 + 3v^2u^4 + v^2u^2 - 1 \\ y_3 &= u(2v^2u^6 + 3v^2u^4 + v^2u^2 + 1) \\ y_4 &= -2u^3v \end{aligned} \tag{37}$$

Now we can substitute these polynomials into (29) and obtain

$$\bar{t}(u,v) = \frac{4uv}{v^4(4u^{12}+12u^{10}+13u^8+6u^6+u^4+1) + v^2(4u^6+10u^4+2u^2)+1}. \tag{38}$$

After lifting this parameterization back into five-dimensional space we obtain the five bivariate polynomials

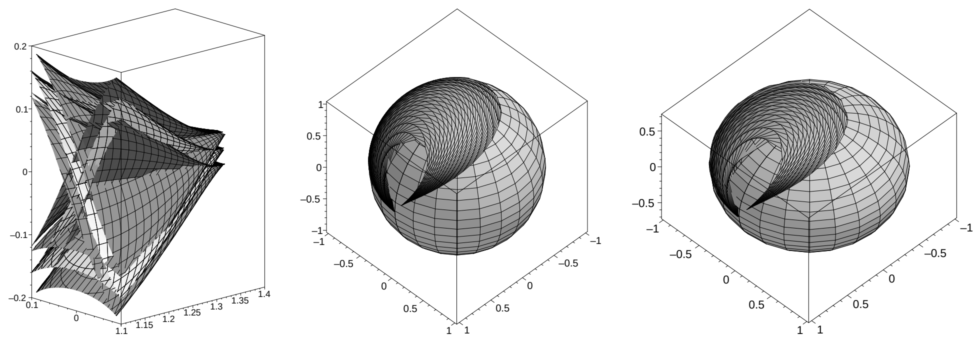
$$\begin{aligned} x_1 &= v^4(4u^{12}+12u^{10}+13u^8+6u^6+u^4) + v^2(4u^6-6u^4-6u^2)+1 \\ x_2 &= 4uv(3v^2u^4+2v^2u^6+v^2u^2-1) \\ x_3 &= 4u^2v(3v^2u^4+2v^2u^6+v^2u^2+1) \\ x_4 &= v^4(4u^{12}+12u^{10}+13u^8+6u^6+u^4) + v^2(4u^6+2u^4+2u^2)+1 \\ x_5 &= v^4(4u^{12}+12u^{10}+13u^8+6u^6+u^4) + v^2(4u^6+10u^4+2u^2)+1 \end{aligned} \tag{39}$$

that satisfy the two identities (25).

The corresponding two parameterizations $\frac{1}{x_4}(x_1, x_2, x_3)$ and $\frac{1}{x_5}(x_1, x_2, x_3)$ of the unit sphere and of the ellipsoid are shown in Fig. 2c,d. In both cases, the parameters u, v vary in the domain $[0, 1.5] \times [0, 1.5]$.

Finally we evaluate the envelope operator (26) and obtain the parameterization $\mathbf{z}(u,v)$ of the surface with the support function h. This parameterization is presented in Table 1.

Example 11. In this example we consider the surface given by the support function $h(\mathbf{x}) = \sqrt{x_1^2 + x_2^2 - x_3^2} + 1$. It is the offset at distance 1 of a one-sheeted hyperboloid of revolution. Applying the parameterization process as in the previous example, we obtain



(a) (b) (c)

Figure 2. The surface of example 10 and its offsets (a), and the simultaneous parameterizations of the sphere and of the ellipsoid (b,c).

Table 1. Parameterization of the surface from Example 10

$$
\begin{aligned}
z_1 = \frac{1}{D}&(1 - 20v^5u^6 + 32v^7u^{20} + 248v^8u^{20} + 132v^6u^{10} + 48v^5u^{12} + 36v^4u^6 + 252v^7u^{14}\\
&+ 4v^7u^8 - 72v^5u^8 + 24v^3u^8 - 12v^3u^6 + 36v^7u^{10} + 48v^5u^{14} + 4vu^2 + 360v^8u^{18} + 4v^6u^6\\
&+ 36v^6u^8 + 8v^2u^6 + 180v^8u^{14} + 144v^7u^{18} + 264v^7u^{16} - 52v^5u^{10} + 252v^6u^{12} + 4v^2u^2\\
&+ 72v^4u^{10} + 16v^8u^{24} + 24v^4u^{12} + 62v^8u^{12} + 62v^4u^8 + 96v^8u^{22} + v^8u^8 + 12v^8u^{10}\\
&+ 6v^4u^4 + 32v^6u^{18} - 20v^3u^4 + 144v^6u^{1}6 + 12v^2u^4 + 264v^6u^{14} + 321v^8u^{16} + 132v^7u^{12})\\
&(1 + 20v^5u^6 - 32v^7u^{20} + 248v^8u^{20} + 132v^6u^{10} - 48v^5u^{12} + 36v^4u^6 - 252v^7u^{14} - 4v^7u^8\\
&+ 72v^5u^8 - 24v^3u^8 + 12v^3u^6 - 36v^7u^{10} - 48v^5u^{14} - 4vu^2 + 360v^8u^{18} + 4v^6u^6 + 8v^2u^6\\
&+ 36v^6u^8 + 180v^8u^{14} - 144v^7u^{18} - 264v^7u^{16} + 52v^5u^{10} + 252v^6u^{12} + 4v^2u^2 + 72v^4u^{10}\\
&+ 16v^8u^{24} + 24v^4u^{12} + 62v^8u^{12} + 62v^4u^8 + 96v^8u^{22} + 12v^8u^{10} + 6v^4u^4 + v^8u^8\\
&+ 32v^6u^{18} + 20v^3u^4 + 144v^6u^{16} + 12v^2u^4 + 264v^6u^{14} + 321v^8u^{16} - 132v^7u^{12})
\end{aligned}
$$

$$
\begin{aligned}
z_2 = \frac{1}{D}&64u^5v^3(1 - 3v^2u^4 - 2v^2u^6 - v^2u^2)(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2)^2\\
&(4v^4u^{12} + 12v^4u^{10} + 13v^4u^8 + 4v^2u^6 + 6v^4u^6 - 6v^2u^4 + v^4u^4 - 6v^2u^2 + 1)
\end{aligned}
$$

$$
\begin{aligned}
z_3 = \frac{1}{D}&4(4v^4u^{1}2 + 12v^4u^{10} + 13v^4u^8 + 4v^2u^6 + 6v^4u^6 - 6v^2u^4 + v^4u^4 - 6v^2u^2 + 1)\\
&u^2v(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2)(1 + 12v^3u^6 + 8v^3u^8 + 4v^2u^6 + 4vu^2 + v^4u^4 + 4v^3u^4\\
&+ 13v^4u^8 + 4v^4u^{1}2 + 12v^4u^{10} + 2v^2u^2 + 2v^2u^4 + 6v^4u^6)(1 - 12v^3u^6 - 8v^3u^8 + 4v^2u^6\\
&- 4vu^2 + 13v^4u^8 + 4v^4u^{12} + 12v^4u^{10} + 2v^2u^2 + 2v^2u^4 + v^4u^4 - 4v^3u^4 + 6v^4u^6)
\end{aligned}
$$

$$
\begin{aligned}
D = &(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2 + 2vu^2)^3(1 + 3v^2u^4 + 2v^2u^6 + v^2u^2 - 2vu^2)^3\\
&(1 + 2v^2u^2 + 10v^2u^4 + 4v^2u^6 + v^4u^4 + 6v^4u^6 + 13v^4u^8 + 12v^4u^{10} + 4v^4u^{12})
\end{aligned}
$$

the following parameterization:

$$z_1 = \frac{1}{D}(2v^4u^{12} - 4v^4u^8 - 4v^2u^6 + 2v^4u^4 - 12v^2u^2 + 2)(-1 + v^2u^6 - v^2u^2)^2$$

$$z_2 = -\frac{1}{D}8uv(-1 + v^2u^6 - v^2u^2)^2(1 + v^2u^6 - v^2u^2)$$

$$z_3 = \frac{1}{D}64u^6v^3(-1 + v^2u^6 - v^2u^2)$$

$$
\begin{aligned}
D = &(v^4u^{12} - 2v^4u^8 - 2v^2u^6 - 8u^4v^2 + v^4u^4 + 2v^2u^2 + 1)\\
&(v^4u^{12} - 2v^4u^8 - 2v^2u^6 + 8u^4v^2 + v^4u^4 + 2v^2u^2 + 1)
\end{aligned}
$$

Although the degree of this surface is $(8, 24)$, the representation is quite compact as the polynomials are very sparse.

We summarize the results of this section in the following Theorem.

Theorem 12. *For a DIAMOND surface $\mathcal{S}$ with $b \neq c$, i.e., for a surface with a support function of the form* (14)*, we obtain a piecewise rational parameterization by combining the result of Algorithm 9 with Lemma 8. If b, c and all other coefficients in the given support function h are rational numbers, then all coefficients of this parameterization are again rational.*

Finally we analyze the case $b = c$, which was excluded so far.

Remark 13. If $b = c = 1$, then we can simply parameterize the unit sphere and choose $x_4 = \pm x_5$. If $1 \neq b = c > 0$ we can solve the problem by swapping the first two coordinates.

The case $b = c < 0$ is more involved. For instance, the offsets of two-sheeted hyperboloids of revolution belong to this case. After stereographic projection and dehomogenization ($x_4 = 1$) we obtain the cubic surface

$$k = (x_1 + b - bx_1)x_2^2 + (x_1 + b - bx_1)x_3^2 + x_1^2 - x_1 = 0 \tag{40}$$

which can be rewritten as

$$r_1 r_2 x_2^2 + r_1 r_2 x_3^2 = r_2^2, \tag{41}$$

with $r_1 = (x_1 + b - bx_1)$ and $r_2 = x_1 - x_1^2$. In order to admit solutions, the factor $r_1 r_2$ has to be positive. This is the case if $x_1 \in]-\infty, 0[$ or $x_1 \in]\frac{-b}{1-b}, 1]$. Here we discuss the first situation. The second one can be treated similarly.

After substituting $x_1 = -t^2$ in (40), we obtain

$$r_1 r_2 = t^6 + t^4 - 2bt^4 - bt^2 - bt^6 = A^2 + B^2 \quad \text{and} \quad r_2 = -t^4 - t^2. \tag{42}$$

with

$$A = t^3\sqrt{1-b} - t\sqrt{-b} \quad \text{and} \quad B = t^2\sqrt{1-b} + t^2\sqrt{-b}. \tag{43}$$

Note that $r_1 r_2$ is now non–negative for all values of t, hence it is possible to represent it as a sum of two squares. The point with coordinates

$$x_1 = -t^2, \quad x_2 = \frac{Ar_2}{A^2 + B^2} \quad \text{and} \quad x_3 = \frac{Br_2}{A^2 + B^2} \tag{44}$$

lies on each of the circles, and it can be used to create a parameterization of the cubic surface (40). Note that this parameterization has coefficients involving certain square roots of the original coefficients, as a decomposition of a polynomial into a sum of squares is needed.

6. Conclusion

Motivated by the analysis of offsets to quadric surfaces, we analyzed a class of surfaces which have a special type of support functions. It was shown that these surfaces admit rational parameterizations. If the given support function involves only coefficients which are rational numbers, then the coefficients of these parameterizations are again rational numbers. In particular this applies to offsets of quadric surfaces.

When applied to offsets of quadric surfaces, our method produces a parameterization of higher degree than the parameterization obtained by Peternell and Pottmann (1998).

The technique described in the present paper provides an alternative approach to offsets of quadric surfaces. The approach is more general, as it can deal with a larger class of surfaces. As a potential advantage, it relies solely on rational operations. In particular, no decomposition of a non-negative polynomial in a sum of squares (and hence no field extension) is needed.

As a possible topic of future work one may look into general rational parameterizations of the cubic surfaces from the previous sections. It can be shown that each rational parameterization of a DIAMOND surface corresponds to a rational parameterization of this cubic. Consequently, one may try to obtain parameterizations of lower degree for DIAMOND surfaces by using other parameterizations of the cubic surfaces.

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