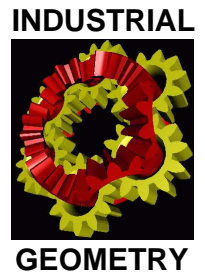


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**A Note on the Convolutions of
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with Quadratic Polynomial (and
Other) Surfaces**

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A Note on the Convolutions of Rational Surfaces with Quadratic Polynomial (and Other) Surfaces

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Abstract. The convolutions of a non-developable quadratic polynomial surface with rational surfaces always admit a rational parameterization. In particular, the offsets of these surfaces can be obtained as convolutions with rational surfaces describing the sphere, and the surfaces can be reparameterized such that they have a linear field of normal vectors.

Lávička and Bastl (2006) proved this result in a more general framework using Groebner bases. Later, Peternell and Odehnal (2006) provided synthetic arguments and proved this observation with the help of the theory of Cremona transformations.

In this note we prove this result by a direct computation. As a byproduct we obtain a simple formula for computing the convolution surfaces. In addition we extend the result to a class of translational surfaces.

Keywords. Rational parameterization, quadratic polynomial surfaces, polynomial Steiner surfaces, convolution, offset.

We consider a segment of a quadratic polynomial surface

$$\mathbf{x} : \Omega_0 \rightarrow \mathbb{R}^3 : (u_1, u_2) \mapsto \mathbf{x}(u_1, u_2) = \sum_{i=0}^2 \sum_{j=0}^{2-i} \mathbf{a}_{ij} u_1^i u_2^j, \quad (1)$$

with certain constant coefficient vectors $\mathbf{a}_{ij} \in \mathbb{R}^3$. We assume that the parameter domain $\Omega_0 \subseteq \mathbb{R}^2$ is chosen such that all points are regular,

$$\forall (u_1, u_2) \in \Omega_0 : \mathbf{x}_1(u_1, u_2) \times \mathbf{x}_2(u_1, u_2) \neq \mathbf{0} \quad (2)$$

where

$$\mathbf{x}_1(u_1, u_2) = 2\mathbf{a}_{20}u_1 + \mathbf{a}_{11}u_2 + \mathbf{a}_{10}, \quad \mathbf{x}_2(u_1, u_2) = \mathbf{a}_{11}u_1 + 2\mathbf{a}_{02}u_2 + \mathbf{a}_{01} \quad (3)$$

are the partial derivative vectors with respect to u_1 and u_2 , and also such that the surface does not contain any parabolic points¹. We define the sets of the

¹The parabolic points of quadratic polynomial surfaces are located along straight lines in the parameter domain, and the direction of the normal along the corresponding parabolic lines on the surface is constant. Indeed, if $\mathbf{x}(0,0)$ is a parabolic point with asymptotic direction $(1,0)$, then all points $\mathbf{a}_{i,j}$ with $j < 2$ are contained in the tangent plane at $\mathbf{x}(0,0)$. Consequently, the parameter line $u_2 = 0$ contains only parabolic points.

non-normalized normals and of the *admissible normals* of the surface segment (1),

$$A_0 = \{\mathbf{x}_1(u_1, u_2) \times \mathbf{x}_2(u_1, u_2) : (u_1, u_2) \in \Omega_0\} \quad \text{and} \quad A = (\mathbb{R} \setminus \{0\}) \cdot A_0, \quad (4)$$

respectively. The latter set is obtained by forming all non-zero multiples of vectors in A_0 .

For any given vector $\mathbf{n} \in A$, the cross product of the partial derivative vectors and \mathbf{n} are linearly dependent at a certain point $(u_1, u_2) \in \Omega_0$,

$$\text{rank}(\mathbf{x}_1(u_1, u_2) \times \mathbf{x}_2(u_1, u_2), \mathbf{n}) < 2, \quad (5)$$

if and only if the two equations

$$\begin{aligned} 0 &= \mathbf{x}_1(u_1, u_2) \cdot \mathbf{n} = 2(\mathbf{a}_{20} \cdot \mathbf{n})u_1 + (\mathbf{a}_{11} \cdot \mathbf{n})u_2 + (\mathbf{a}_{10} \cdot \mathbf{n}), \text{ and} \\ 0 &= \mathbf{x}_2(u_1, u_2) \cdot \mathbf{n} = (\mathbf{a}_{11} \cdot \mathbf{n})u_1 + 2(\mathbf{a}_{02} \cdot \mathbf{n})u_2 + (\mathbf{a}_{01} \cdot \mathbf{n}) \end{aligned} \quad (6)$$

are simultaneously satisfied, where the dot \cdot represents the usual inner product in \mathbb{R}^3 . We solve them for (u_1, u_2) using Cramer's rule and obtain the *convolution mapping*

$$\mathcal{C} : A \rightarrow \mathbb{R}^2 : \quad \mathbf{n} \mapsto \left(\frac{N_1(\mathbf{n})}{D(\mathbf{n})}, \frac{N_2(\mathbf{n})}{D(\mathbf{n})} \right) \quad (7)$$

where

$$N_1(\mathbf{n}) = \begin{vmatrix} -(\mathbf{a}_{10} \cdot \mathbf{n}) & (\mathbf{a}_{11} \cdot \mathbf{n}) \\ -(\mathbf{a}_{01} \cdot \mathbf{n}) & 2(\mathbf{a}_{02} \cdot \mathbf{n}) \end{vmatrix}, \quad N_2(\mathbf{n}) = \begin{vmatrix} 2(\mathbf{a}_{20} \cdot \mathbf{n}) & -(\mathbf{a}_{10} \cdot \mathbf{n}) \\ (\mathbf{a}_{11} \cdot \mathbf{n}) & -(\mathbf{a}_{01} \cdot \mathbf{n}) \end{vmatrix} \quad (8)$$

and

$$D(\mathbf{n}) = \begin{vmatrix} 2(\mathbf{a}_{20} \cdot \mathbf{n}) & (\mathbf{a}_{11} \cdot \mathbf{n}) \\ (\mathbf{a}_{11} \cdot \mathbf{n}) & 2(\mathbf{a}_{02} \cdot \mathbf{n}) \end{vmatrix} = \mathbf{n}^\top (4 \mathbf{a}_{20} \mathbf{a}_{02}^\top - \mathbf{a}_{11} \mathbf{a}_{11}^\top) \mathbf{n}. \quad (9)$$

If \mathbf{n} is the unit normal at $\mathbf{x}(u_1, u_2)$, then $D(\mathbf{n})$ is the determinant of the second fundamental form of the surface at this point. Consequently, $D(\mathbf{n}) \neq 0$ for all $\mathbf{n} \in A$, since the domain Ω_0 was assumed to be chosen such that the surface segment does not contain any parabolic points. Hence, *the convolution mapping \mathcal{C} introduced in (7) is well-defined for all $\mathbf{n} \in A$ and it describes a bijection between A_0 and Ω_0 .*

For any other rational surface segment

$$\mathbf{y} : \Omega_1 \rightarrow \mathbb{R}^3 : \quad (v_1, v_2) \mapsto \mathbf{y}(v_1, v_2) \quad (10)$$

with domain $\Omega_1 \subseteq \mathbb{R}^2$ we consider the non-normalized normals

$$\mathbf{N} : \Omega_1 \rightarrow \mathbb{R}^3 : \quad (v_1, v_2) \mapsto \mathbf{y}_1(v_1, v_2) \times \mathbf{y}_2(v_1, v_2), \quad (11)$$

where \mathbf{y}_1 and \mathbf{y}_2 are the partial derivative vectors with respect to v_1 and v_2 , respectively. We assume that $\mathbf{N}(\Omega_1) \subseteq A$, i.e., the domain of \mathbf{y} is to be chosen

such that the non-normalized normals of \mathbf{y} are contained in the set of admissible normals A .

The mapping

$$\Omega_1 \rightarrow \mathbb{R}^3 : (v_1, v_2) \mapsto \mathbf{y}(v_1, v_2) + (\mathbf{N} \circ \mathcal{C} \circ \mathbf{x})(v_1, v_2) \quad (12)$$

then defines a rational parameterization of the so-called *convolution surface* of \mathbf{x} and \mathbf{y} . This surface consists of all points $\mathbf{x}(u_1, u_2) + \mathbf{y}(v_1, v_2)$ where the non-normalized normals $\mathbf{N}(v_1, v_2)$ and $\mathbf{x}_1(u_1, u_2) \times \mathbf{x}_2(u_1, u_2)$ are linearly dependent. Consequently, *convolution surfaces of non-developable quadratic polynomial surfaces with other rational surfaces are again rational.*

If we choose a linearly parameterized field of normals,

$$\mathbf{L} : \Omega_2 \rightarrow \mathbb{R}^3 : (w_1, w_2) \mapsto \mathbf{b}_{10}w_1 + \mathbf{b}_{01}w_2 + \mathbf{b}_{00} \quad (13)$$

representing a segment of a plane which does not pass through the origin, where the domain Ω_2 is chosen such that $\mathbf{L}(\Omega_2) \subseteq A$, then the mapping

$$\Omega_2 \rightarrow \mathbb{R}^3 : (w_1, w_2) \mapsto (\mathbf{L} \circ \mathcal{C} \circ \mathbf{x})(w_1, w_2) \quad (14)$$

defines a segment of an *LN surface* in the sense of Definition 1 of (Sampoli, Peternell and Jüttler, 2006). Hence, *any non-developable quadratic polynomial surface can be reparameterized as a surface with a linear field of normal vectors* (LN surface).

Finally we note that these results can be extended to special translational surfaces,

$$\mathbf{x}(u_1, u_2) = \mathbf{p}(u_1) + \mathbf{q}(u_2) \quad (15)$$

where the generating rational curves satisfy

$$\dot{\mathbf{p}}(u_1) = \phi(u_1) \mathbf{k}(u_1) \quad \text{and} \quad \dot{\mathbf{q}}(u_2) = \psi(u_2) \mathbf{l}(u_2) \quad (16)$$

with arbitrary rational functions ϕ, ψ and linear vector-valued polynomials \mathbf{k}, \mathbf{l} .

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