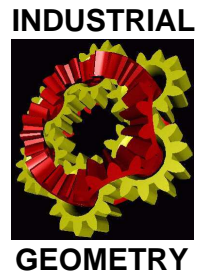


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Evolving Four-Bars for Optimal Synthesis

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EVOLVING FOUR-BARS FOR OPTIMAL SYNTHESIS

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ABSTRACT- We present an evolution-based method for optimal mechanism synthesis. It is based on the embedding of the Euclidean motion group in the space of affine displacements upon which an object-oriented Euclidean metric is imposed. This Euclidean structure allows the use of curve and surface evolution techniques from computer aided design and image processing. We demonstrate the algorithm by synthesizing planar four-bar mechanisms and we show how to adapt it so that the resulting four-bar is free of circuit defects.

KEYWORDS: optimal mechanism synthesis, curve and surface evolution, four-bar mechanism, circuit defect

1. INTRODUCTION

Optimal mechanism synthesis means finding a mechanism (type, dimension, position of base and end-effector) that *approximately* performs a certain task. It is a well-established field in mechanism science that evolved from the natural need for a more extensive but less accurate task description. In this article we present a novel approach to optimal motion generation.

There exists a wealth of literature on optimal mechanism synthesis and we confine ourselves to just a few references on optimal motion generation. The maybe most direct approaches use numerical techniques for solving an over-constrained system of equations (Zhang, Zhou and Ye 2000; Hayes, Luu and Chang 2004; Smaili and Diab 2005). Mixed numerical and graphical methods for optimal four-bar synthesis have been presented by Yao and Angeles 2000. They allow identification of all critical points of the underlying optimization problem. Finally, we can distinguish genetic algorithms (Cabrera, Simon and Prado 2002; Kunjur and Krishnamurty 1995; Zhou and Cheung 2001) and methods based on evolution techniques, for example Shiakolas, Koladiya and Kebrle 2002, 2005.

Our approach fits into the last category. Its main distinction from other optimal synthesis schemes is the use of an object-oriented metric that makes the design space Euclidean. This allows the advantageous use of geometric techniques from computer aided design and image processing. In particular,

we perform closest point computation and curve/surface evolution in this space. Evolution techniques permit great flexibility and control over the mechanism shape during the evolution and can be used to avoid mechanism defects. Moreover, the formulation of the synthesis problem within this framework is independent of the fixed coordinate frame. While our concept is general and can, at least in principle, be used for synthesizing many mechanisms of diverse type and shape, we confine ourselves to the synthesis of planar four-bars. Extensions to further mechanism types will be discussed in a subsequent publication.

In Section 2 we introduce the object-oriented metric used in our approach. It is followed by an outline of a general framework for curve and surface fitting via evolution in Section 3. In Section 4 we explain how to formulate a four-bar synthesis problem within this framework and present a first example. In Section 5, we discuss how to incorporate additional constraints in order to avoid unwanted defects of the synthesized four-bars and in Section 6 we conclude the paper with directions for future research.

2. A METRIC FOR AFFINE DISPLACEMENTS

Measuring the “distance” between two displacements is a fundamental problem of mechanism science with many important applications, in particular in mechanism synthesis. Among the various available concepts (see for example Larochelle, Murray and Angeles 2004; Angeles 2006) we choose an approach which leads to an embedding of the Euclidean motion group into the space of affine displacements that is equipped with an *object-oriented* Euclidean metric (Belta and Kumar 2002; Chirikjian and Zhou 1998). This metric has already been used for motion design (Hofer, Pottmann and Ravani 2004; Hofer and Pottmann 2004), for computing generalized penetration depths (Nawratil, Pottmann and Ravani 2007), and for defining performance indices of robots (Nawratil 2007), but apparently not in mechanism design.

The distance concept is based on the choice of a number of feature points $m_i \in \mathbb{R}^2$. These feature points define a positive definite scalar product $\langle \alpha, \beta \rangle = \sum \langle \alpha(m_i), \beta(m_i) \rangle$ and an associated squared distance $\text{dist}^2(\alpha, \beta) = \sum \|\alpha(m_i) - \beta(m_i)\|^2$ between α and β that depends only on the barycenter of the feature points and on their inertia matrix. It endows the space $\text{GA}^+(2)$ of affine displacements with a *Euclidean metric*. The space $\text{GA}^+(2)$ contains the Euclidean motion group $\text{SE}(2)$ as a sub-manifold of dimension three. The feature points m_i can be replaced by an arbitrary mass distribution μ on \mathbb{R}^2 . In this case the formulas for inner product and distance are expressed in terms of integrals (Chirikjian and Zhou 1998). The inner product defined by the

feature point m_i (or by the mass distribution μ) is also generated by four suitable chosen points of equal mass. Hence, the computation of inner product and distance is very efficient. Extensions of this distance concept to higher dimensions are straightforward.

3. EVOLUTION-BASED FITTING OF ALGEBRAIC VARIETIES

Now we describe a general concept for fitting implicitly defined varieties to a number of prescribed feature points (see also Aigner and Jüttler 2007 and Aigner, Šír and Jüttler 2007). Its relation to optimal synthesis of four-bars will be explained in the next section.

Consider a system $F_1(x), \dots, F_m(x)$ of polynomials and its zero set $C \subset \mathbb{R}^d$. We assume that the coefficients of F_i depend on a vector $s = (s_1, \dots, s_p)$ of *shape parameters*. The vector s of shape parameters is to be determined such that the zero set C of the polynomials F_i approximates a set of target points q_0, \dots, q_n . By p_j we denote the point on C that minimizes the distance to q_j and by ∇F_i we denote the gradient of F_i with respect to x .

Now we view s as depending on a time-variable t , that is $s = s(t)$. As $s(t)$ varies in time, the points $p_j = p_j(t)$ move along certain trajectories. Because of $F_i(p_j(t), s(t)) \equiv 0$ we also have

$$\frac{d}{dt}F_i(p_j(t), s(t)) = \langle \nabla F_i, \dot{p}_j \rangle + \sum_{l=1}^p \frac{\partial F_i}{\partial s_l} \dot{s}_l \equiv 0, \quad (1)$$

where the derivatives of F_i are evaluated at $(p_j(t), s_j(t))$ and the dot denotes derivatives with respect to t . Hence, \dot{p}_j is the velocity of the closest point p_j . By $\{b_{j1}, \dots, b_{jm}\}$ we denote an orthonormal basis of the normal space of C in p_j . Ideally, the change of $s(t)$ should be such that $q_j - p_j = \dot{p}_j$, that is,

$$\langle q_j - p_j, b_{jk} \rangle = \langle \dot{p}_j, b_{jk} \rangle \quad \text{for } j = 0, \dots, n \text{ and } k = 1, \dots, m. \quad (2)$$

There exist coefficients β_{jki} such that $b_{jk} = \sum_{i=1}^m \beta_{jki} \nabla F_i$. In view of (1), this yields

$$\langle \dot{p}_j, b_{jk} \rangle = \sum_{i=1}^m \beta_{jki} \langle \dot{p}_j, \nabla F_i \rangle = - \sum_{i=1}^m \beta_{jki} \sum_{l=1}^p \frac{\partial F_i}{\partial s_l} \dot{s}_l \quad (3)$$

Hence, (2) leads to an over-constrained system of linear equations in the unknowns \dot{s}_j that can be solved in least-squares sense. The result is independent of the choice of the basis vectors b_{jk} . Following the thus defined flow in the manifold of shape parameters leads to a locally optimal solution. It can be shown that the numerical integration of the trajectories via explicit Euler steps is equivalent to a Gauss-Newton method for non-linear least-squares optimization.

4. A CONCEPT FOR OPTIMAL FOUR-BAR SYNTHESIS

The evolution technique for fitting algebraic varieties to points as described in Section 3 can be used for optimal mechanism synthesis. In principle it is applicable for arbitrary mechanisms (planar, spherical, spatial, serial, or parallel). We are going to describe the concept for the case of planar four-bar synthesis. Extensions to other classes of mechanisms are postponed to a further publication.

We model a four-bar motion via two *circle constraints*. For $i = 1, 2$ the point with coordinates (a_i, b_i) in the moving system is constraint to a circle with center (ξ_i, η_i) and radius ρ_i . Then the synthesis problems relates to the concept of Section 3 as follows:

- The target points q_0, \dots, q_n are $n + 1$ prescribed poses, viewed as points on the Euclidean motion group $SE(2) \subset GA^+(2)$.
- The vector s of shape parameters has the ten entries $a_1, b_1, \xi_1, \eta_1, \rho_1, a_2, b_2, \xi_2, \eta_2$, and ρ_2 .
- Three of the five implicit algebraic equations F_1, \dots, F_5 define $SE(2)$ (plus the irrelevant orientation-reversing Euclidean displacements) as submanifold of $GA^+(2)$. The two remaining are the circle constraints $\|\alpha(a_i, b_i) - (\xi_i, \eta_i)\|^2 = \rho_i^2$.
- All distances and angles in \mathbb{R}^6 are measured using the inner product $\langle \cdot, \cdot \rangle$, which corresponds to the feature-point dependent metric, as described in Section 2. Also, the gradient vectors are computed with respect to that metric.

As a first example we fit a four-bar to eleven prescribed poses, proposed by J. Michael McCarthy on the occasion of the ASME Design Engineering Technical Conference in Montreal in 2002 (see also Chen and Angeles 2007). Figure 1 shows these poses, the initial configuration, and the results after 15 and 30 iterations. In each picture we display the target poses (dark), the closest poses of the current four-bar motion (white), and the four-bar mechanism in a configuration to reach the third of the prescribed poses. As can be seen, the prescribed poses are particularly amenable to a four-bar approximation.

In all examples of this article we choose the mass distribution by defining a set M of feature points that roughly represents the shape of the end-effector as depicted in Figure 1. The total mass distribution is the union of all images of M under the prescribed poses.

In order to control the step-size in the evolution sequence, we specify a certain maximum allowed displacement H for the closest points p_j . The squared length of the displacement vector of p_j approximately equals $\delta_j^2 = \sum_{k=1}^m \langle \dot{p}_j, b_{jk} \rangle^2$ and can be computed via Equation (3). The allowed step size h for the current step is obtained as $h = H \min\{\delta_j^{-1}\}$. Furthermore, we decrease

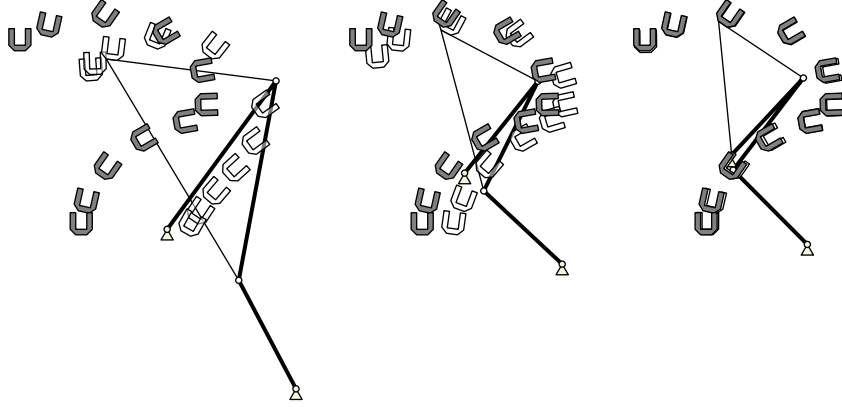


Figure 1: Solution to McCarthy's eleven poses (from left to right: initial four-bar, after 15 iterations, after 30 iterations)

the value of H once the maximal expected change of the closest points is smaller than H in order to avoid forcing a large step away from an already good configuration.

Currently, the most time-consuming part of the evolution procedure consists in the computation of the closest points on the constraint variety in $GA^+(2)$. This is equivalent to computing the closest pose on a four-bar motion to a given pose. In our implementation we compute this points via standard constrained optimization routines. This is not completely reliable and we have no guarantee that we really end up in the global minimum. We found, however, that using the closest points from the preceding iteration step as starting values for the new closest points performs reasonably well.

5. PENALTY FUNCTIONS

In this section we show how to integrate penalty functions into our evolution framework. We discuss how to avoid singular four-bars and use this technique to produce optimal solutions that are free of circuit defects. Denote the base length of the four-bar by g and the coupler length by h . The lengths of driver and follower are the shape parameters ρ_1 and ρ_2 , respectively, while g and h are simple functions of the remaining shape parameters. Then a four-bar is singular (folding) if and only if one of the functions

$$C_1 = \frac{(\rho_1 + \rho_2 - g - h)^2}{(\rho_1 + \rho_2 + g + h)^2}, \quad C_2 = \frac{(\rho_1 - \rho_2 + g - h)^2}{(\rho_1 + \rho_2 + g + h)^2}, \quad C_3 = \frac{(\rho_1 - \rho_2 - g + h)^2}{(\rho_1 + \rho_2 + g + h)^2} \quad (4)$$

vanishes. For each of the functions C_i we choose a minimal permissible value δ_i , and an offset σ_i . Using these values, we construct a smooth penalty

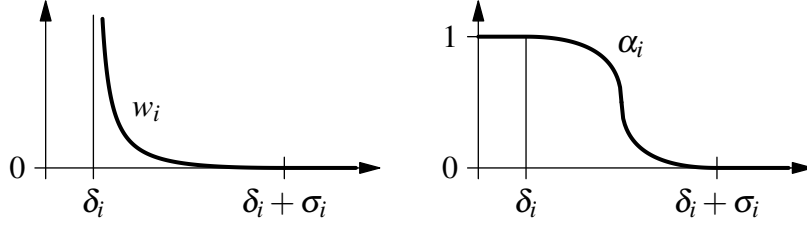


Figure 2: Penalty function w_i and activator function α_i

function w_i and an activator function α_i as depicted in Figure 2. If C_i falls within $[\delta_i, \delta_i + \sigma_i]$, then the evolution pushes the shape parameters away from the zero set of C_i . This is achieved by adding the equation

$$\alpha(C_i) \cdot \left(\sum_{l=1}^{10} \frac{\partial C_i}{\partial s_l} \dot{s}_l - w_i(C_i) \right) = 0, \quad (5)$$

which is linear in \dot{s} , to the over-constrained system obtained from Equation (3).

A four-bar mechanism is called circuit defective with respect to a set of poses, if it cannot attain the poses within the same continuous range of motion. It needs to be disassembled and re-configured in a different way in order to reach all poses. In our case, the poses are only attained approximately. Therefore, we say that a four-bar exhibits a *circuit defect*, if it is circuit defective with respect to the closest poses.

Assume now that the optimal four-bar obtained via evolution is circuit defective but the initial curve that evolves towards the target was not. Since the closest points $p_i(t)$ move smoothly during the evolution process, this can only happen if the topology of the four-bar motion – which is seen as a curve on $SE(2) \subset GA^+(2)$ – changes to two disconnected components (either from one or two components, or from two components via one component back to two components). At the transition between any two stages is a folding four-bar. Therefore, four-bars with circuit defect can be prevented by avoiding folding four-bars. A similar technique can also avoid Grashof defects and mechanism with a bad transmission ratio.

An example avoiding a circuit defect is shown in Figure 3. The five prescribed poses permit precisely one exact solution that suffers, however, from a circuit defect. The direct application of our algorithm (left-hand side) produces this defective mechanism. The picture on the right-hand side of Figure 3 shows an optimal solution, generated from the same initial values, that avoids the assembly mode defect.

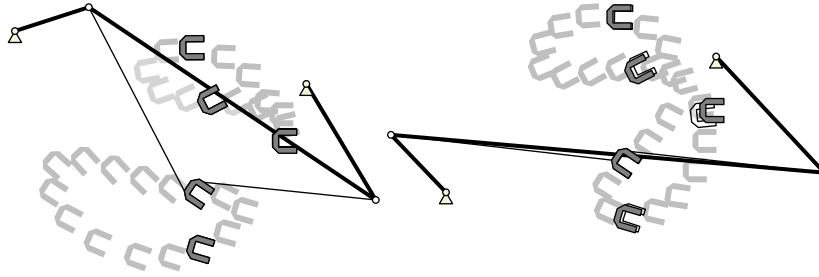


Figure 3: Avoiding solutions with circuit defect.

6. CONCLUSION

We presented an algorithm for optimal mechanism synthesis (motion generation). In this paper we discussed the synthesis of a planar four-bars. The extension to other classes of mechanisms and incorporation of further defect-avoiding constraints is subject of ongoing research.

The evolution viewpoint to numerical optimization – as described in this paper – provides several advantages, including the connection to well-established techniques from image processing, the availability of a geometrically motivated step-size control, the possibility to add constraints via penalty functions, and the invariance with respect to the specific choice of the shape parameters (see Aigner and Jüttler 2008). In addition our method provides a geometrically significant solution, due to left-invariance of the used metric and the repeated closest point computation.

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