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O. Aichholzer, T. Hackl, M. Hoffmann, C. Huemer, F. Santos, B. Speckmann and B. Vogtenhuber

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Maximizing Maximal Angles for Plane Straight Line Graphs

Oswin Aichholzer* Thomas Hackl* Michael Hoffmann† Clemens Huemer‡ Francisco Santos§
Bettina Speckmann¶ Birgit Vogtenhuber*

Abstract

Let $G = (S, E)$ be a plane straight line graph on a finite point set $S \subset \mathbb{R}^2$ in general position. For a point $p \in S$ let the *maximum incident angle* of p in G be the maximum angle between any two edges of G that appear consecutively in the circular order of the edges incident to p. A plane straight line graph is called φ-*open* if each vertex has an incident angle of size at least φ. In this paper we study the following type of question: What is the maximum angle φ such that for any finite set $S \subset \mathbb{R}^2$ of points in general position we can find a graph from a certain class of graphs on S that is φ-open? In particular, we consider the classes of triangulations, spanning trees, and paths on S and give tight bounds in all but one cases.

1 Introduction

Conditions on angles in plane straight-line graphs have been studied extensively in discrete and computational geometry. It is well known that Delaunay triangulations maximize the minimum angle over all triangulations, and that in a (Euclidean) minimum weight spanning tree each angle is at least $\frac{\pi}{3}$. In this paper we address the fundamental combinatorial question, what is the maximum value α such that for each finite point set in general position there exists a plane straight-line graph (of a certain type) where each vertex has an incident angle of size at least α. We present bounds on this value for three classes of graphs: spanning paths (general and bounded degree), spanning trees, and triangulations. Most of the bounds we give are tight. In order to show that, we describe families of point sets for which no graph from the respective class can achieve a greater incident angle at all vertices.

*Institute for Software Technology, Graz University of Technology, [oaich|thackl|bvogt]@ist.tugraz.at. Supported by the Austrian FWF Joint Research Project 'Industrial Geometry' S9205-N12.

†Institute for Theoretical Computer Science, ETH Zürich, hoffmann@inf.ethz.ch

‡Departament de Matemática Aplicada II, Universitat Politécnica de Catalunya, Huemer.Clemens@upc.edu. Partially supported by projects MEC MTM2006-01267 and Gen. Cat. 2005SGR00692.

§Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, francisco.santos@unican.es. Partially supported by grant MTM2005-08618-C02-02 of the Spanish Ministry of Education and Science.

¶Department of Mathematics and Computer Science, TU Eindhoven, speckman@win.tue.nl

Background. Our motivation for this research stems from the investigation of "pseudo-triangulations", a straight-line framework which, apart from deep combinatorial properties, has applications in motion planning, collision detection, ray shooting and visibility; see [1, 9, 10, 12, 13] and references therein. Pseudo-triangulations with a minimum number of pseudo-triangles (among all pseudo-triangulations for a given point set) are called *minimum* (or *pointed*) pseudo-triangulations. They can be characterized as plane straight-line graphs where each vertex has an incident angle greater than π. Furthermore, the number of edges in a minimum pseudo-triangulation is maximal, in the sense that the addition of any edge produces an edge-crossing or negates the angle condition.

In comparison to these properties, we consider connected plane straight-line graphs where each vertex has an incident angle α – to be maximized – and the number of edges is minimal (spanning trees) and the vertex degree is bounded (spanning trees of bounded degree and spanning paths, respectively). We further show that any planar point set has a triangulation in which each vertex has an incident angle of at least $\frac{2\pi}{3}$. Observe that perfect matchings can be described as plane straight-line graphs where each vertex has an incident angle of 2π and the number of edges is maximal.

Related Work. There is a vast literature on triangulations that are optimal according to certain criteria. Similar to Delaunay triangulations which maximize the smallest angle over all triangulations for a point set, farthest point Delaunay triangulations minimize the smallest angle over all triangulations for a convex polygon [6]. If all angles in a triangulation are $\geq \frac{\pi}{6}$, then it contains the relative neighborhood graph as a subgraph [11]. The relative neighborhood graph for a point set connects any pair of points which are mutually closest to each other (among all points from the set). Edelsbrunner et al. [7] showed how to construct a triangulation that minimizes the maximum angle among all triangulations for a set of n points in $O(n^2 \log n)$ time.

In applications where small angles have to be avoided by all means, a Delaunay triangulation may not be sufficient in spite of its optimality because even there arbitrarily small angles can occur. By adding so-called Steiner points one can construct a triangulation on a superset of the original points in which there is some absolute lower bound on the size of the smallest angle [4]. Dai et al. [5] describe several heuristics to construct minimum weight triangulations (triangulations which minimize the total sum of edge lengths) subject to absolute lower or upper bounds on the occurring angles.

Spanning cycles with angle constraints can be regarded as a variation of the traveling salesman problem. Fekete and Woeginger [8] showed that if the cycle may cross itself then any set of at least five points admits a locally convex tour, that is, a tour in which the angle between any three consecutive points is positive. Aggarwal et al. [2] prove that finding a spanning cycle for a point set which has minimal total angle cost is NP-hard, where the angle cost is defined as the sum of direction changes at the points.

Regarding spanning paths, it has been conjectured that each planar point set admits a spanning path with minimum angle at least $\frac{\pi}{6}$ [8]; recently, a lower bound of $\frac{\pi}{9}$ has been presented [3].

Definitions and Notation. Let $S \subset \mathbb{R}^2$ be a finite set of points in general position, that is, no three points of S are collinear. In this paper we consider plane straight line graphs $G = (S, E)$ on S. The vertices of G are precisely the points in S, the edges of G are straight line segments that connect two points in S, and two edges of G do not intersect except possibly at their endpoints. For a point $p \in S$ the *maximum incident angle* $\mathrm{op}_G(p)$ of p in G is the maximum angle between any two edges of G that appear consecutively in the circular order of the edges incident to p. For a vertex $p \in S$ of degree at most one we set $\mathrm{op}_G(p) = 2\pi$. We also refer to $\mathrm{op}_G(p)$ as the *openness* of p in G and call $p \in S$ *φ-open* in G for some angle φ if $\mathrm{op}_G(p) \geq \varphi$. Consider, for example, the graph depicted in Figure 1. The point p has four incident edges in G and, therefore, four incident angles. Its openness is $\mathrm{op}_G(p) = \alpha$. The point q has only one incident angle and correspondingly $\mathrm{op}_G(q) = 2\pi$.

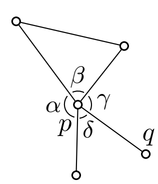

Figure 1: Incident angles of p.

Similarly we define the *openness* of a plane straight line graph $G = (S, E)$ as $\mathrm{op}(G) = \min_{p \in S} \mathrm{op}_G(p)$ and call G *φ-open* for some angle φ if $\mathrm{op}(G) \geq \varphi$. In other words, a graph is φ-open if and only if every vertex has an incident angle of size at least φ. The *openness* of a class $\mathcal{G}$ of graphs is the supremum over all angles φ such that for every finite point set $S \subset \mathbb{R}^2$ in general position there exists a φ-open connected plane straight line graph G on S and G is an embedding of some graph from $\mathcal{G}$. For example, the openness of minimum pseudo-triangulations is π.

Observe that without the general position assumption many of the questions become trivial because for a set of collinear points the non-crossing spanning tree is unique – the path that connects them along the line – and its interior points have no incident angle greater than π.

Let a, b, and c be three points in the plane that are not collinear. With $\angle abc$ we denote the counterclockwise angle between the segment (b, a) and the segment (b, c) at b.

Results. In this paper we study the openness of several well known classes of plane straight line graphs, such as triangulations ($\frac{2\pi}{3}$, Section 2), spanning trees (Section 3) in general ($\frac{5\pi}{3}$) and with maximum degree three ($\frac{3\pi}{2}$), and spanning paths ($\frac{3\pi}{2}$ for sets in convex position, Section 4).

2 Triangulations

Theorem 1 *Every finite point set in general position in the plane has a triangulation that is $\frac{2\pi}{3}$-open and this is the best possible bound.*

Proof. Consider a point set $S \subset \mathbb{R}^2$ in general position. Clearly, $\mathrm{op}_G(p) > \pi$ for every point $p \in \mathrm{CH}(S)$ and every plane straight line graph G on S. We recursively construct a $\frac{2\pi}{3}$-open triangulation T of S by first triangulating $\mathrm{CH}(S)$; every recursive subproblem consists of a point set with a triangular convex hull.

Let S be a point set with a triangular convex hull and denote the three points of $\mathrm{CH}(S)$ with a, b, and c. If S has no interior points, then we are done. Otherwise, let a', b' and c' be (not necessarily distinct) interior points of S such that the triangles $\Delta a'bc$, $\Delta ab'c$ and $\Delta abc'$ are empty (see Figure 2). Since the sum of the six exterior angles of the hexagon $bc'ab'ca'$ equals 8π, the sum of the three angels $\angle ac'b$, $\angle ba'c$, and $\angle cb'a$ is at least 2π. In particular, one of them, say $\angle cb'a$, is at least $2\pi/3$. We then recurse on the two subsets of S that have $\Delta bb'c$ and $\Delta ab'b$ as their respective convex hulls.

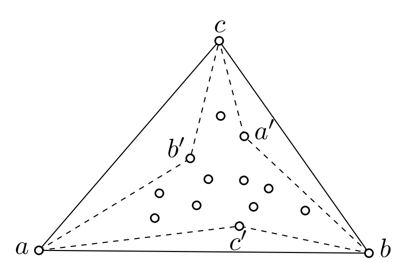

Figure 2: Constructing a $\frac{2\pi}{3}$-open triangulation.

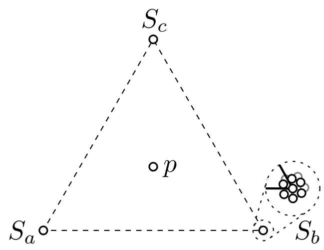

Figure 3: The openness of triangulations of this point set approaches $\frac{2\pi}{3}$.

The upper bound is attained by a set S of n points as depicted in Figure 3. S consists of a point p and of three sets S_a, S_b, and S_c that each contain $\frac{n-1}{3}$ points. S_a, S_b, and S_c are placed at the vertices of an equilateral triangle Δ and p is placed at the barycenter of Δ. Any triangulation T of S must connect p with at least one point of each of S_a, S_b, and S_c and hence $\text{op}_T(p)$ approaches $\frac{2\pi}{3}$ arbitrarily close. □

3 Spanning Trees

In this section we give tight bounds on the φ-openness of two basic types of spanning trees, namely general spanning trees and spanning trees with bounded vertex degree. Consider a point set $S \subset \mathbb{R}^2$ in general position and let p and q be two arbitrary points of S. Assume w.l.o.g. that p has smaller x-coordinate than q. Let l_p and l_q denote the lines through p and q that are perpendicular to the edge (p, q). We refer to the region bounded by l_p and l_q as the *orthogonal slab* of (p, q).

Observation 1 *Assume that $r \in S \setminus \{p, q\}$ lies in the orthogonal slab of (p, q) and above (p, q). Then $\angle qpr \leq \frac{\pi}{2}$ and $\angle rqp \leq \frac{\pi}{2}$. A symmetric observation holds if r lies below (p, q).*

Recall that the diameter of a point set is a pair of points that are furthest away from each other. Let a and b define the diameter of S and assume w.l.o.g. that a has a smaller x-coordinate than b. Clearly, all points in $S \setminus \{a, b\}$ lie in the orthogonal slab of (a, b).

Observation 2 *Assume that $r \in S \setminus \{a, b\}$ lies above a diametrical segment (a, b) for S. Then $\angle arb \geq \frac{\pi}{3}$ and hence at least one of the angles $\angle bar$ and $\angle rba$ is at most $\frac{\pi}{3}$. A symmetric observation holds if r lies below (a, b).*

These two simple observations can be used to obtain the following results on spanning trees.

Theorem 2 *Every finite point set in general position in the plane has a spanning tree that is $\frac{5\pi}{3}$-open, and this bound is tight.*

Theorem 3 *Let $S \subset \mathbb{R}^2$ be a set of n points in general position. There exists a $\frac{3\pi}{2}$-open spanning tree T of S such that every point from S has vertex degree at most 3 in T. The angle bound is best possible, even for the much broader class of spanning trees of vertex degree at most $n - 2$.*

Both proofs for the above theorems are based on an extensive case analysis. Therefore we omit them in this extended abstract. The interested reader can find all details in the full version of the paper or in [14].

4 Spanning Paths

For spanning paths, the upper bound for trees with bounded vertex degree can be applied as well. The resulting bound of $\frac{3\pi}{2}$ is tight for points in convex position, even in a very strong sense: There exists a $\frac{3\pi}{2}$-open spanning path starting from any point. We also give examples showing that our construction cannot be extended to general point sets.

4.1 Point Sets in Convex Position

Consider a set $S \subset \mathbb{R}^2$ of n points in convex position. We can construct a spanning path for S by starting at an arbitrary point $p \in S$ and recursively taking one of the tangents from p to $\text{CH}(S \setminus \{p\})$. As long as $|S| > 2$, there are two tangents from p to $\text{CH}(S \setminus \{p\})$: the left tangent is the oriented line t_ℓ through p and a point from $p_\ell \in S \setminus \{p\}$ (oriented in direction from p to p_ℓ) such that no point from S is to the left of t_ℓ. Similarly, the right tangent is the oriented line t_r through p and a point from $p_r \in S \setminus \{p\}$ (oriented in direction from p to p_r) such that no point from S is to the right of t_r. If we take the left and the right tangent alternately, we call the resulting path a *zigzag* path for S.

Theorem 4 *Every finite point set in convex position in the plane admits a spanning path that is $\frac{3\pi}{2}$-open, and this bound is best possible.*

In the full version of the paper, we present two different proofs for this theorem, an existential proof using counting arguments and a constructive proof. In addition, the latter provides the following stronger statement.

Corollary 5 *For any finite set $S \subset \mathbb{R}^2$ of points in convex position and any $p \in S$ there exists a $\frac{3\pi}{2}$-open spanning path for S which has p as an endpoint.*

4.2 General Point Sets

So far we have not been able to generalize the results of Theorem 4 and Corollary 5 to general point sets. In this section we present a few examples to indicate where the difficulties lie.

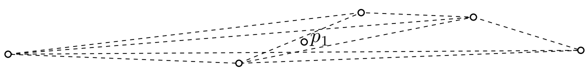

Figure 4: Starting at interior point p_1 results in an at most $(\pi + \varepsilon)$-open spanning path.

Figure 4 depicts a configuration where any spanning path starting at the interior point p_1 is at most $(\pi + \varepsilon)$-open. Figure 5 shows a configuration that has a similar property. Here point p_5 is positioned arbitrarily far to the left and $\beta = \frac{\pi}{3}$. If we require the edge (p_1, p_2) to be part of the spanning path, then we can construct at most a $\left(\frac{4\pi}{3} + \varepsilon\right)$-open spanning path.

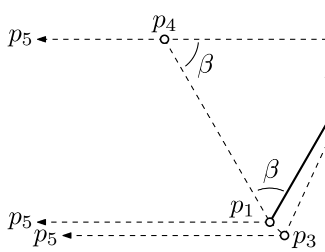

Figure 5: If edge (p_1, p_2) is forced we get at most a $\left(\frac{4\pi}{3} + \varepsilon\right)$-open spanning path.

Both examples show that, whatever approach is used to generate a spanning path, we have to be careful when forcing points or edges to play a specific role in the construction. Especially starting at a fixed interior point has to be avoided.

A direct generalization of the constructive approach for convex sets would be a path which starts at a given extreme point and recursively continues only along tangents to the remaining point set. But there exist examples where this approach generates an at most $(\pi + \varepsilon)$-open spanning path. Details on this construction and the examples presented above, as well as a large variety of much more involved approaches can be found in [14].

On the other hand, and despite the above presented constructions, we have not been able to provide a single point set, which does not contain a $\frac{3\pi}{2}$-open spanning path. To the contrary, computational investigations on several billion random point sets (in the range of $4 \leq n \leq 20$ points) provided for each set a $\frac{3\pi}{2}$-open spanning path, even if we required the path to start with a prefixed extreme point. Thus we conclude this section with the following conjecture.

Conjecture 1 *Spanning paths for general point sets are $\frac{3\pi}{2}$-open.*

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