On general approach to optimal parameterization of polynomial curves

Michael S. Floater Emil Žagar*

CMA and IFI, University of Oslo
IMFM and FMF, University of Ljubljana

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Main problem

- Standard problem in CAGD:
  - Points $\boldsymbol{T}_j \in \mathbb{R}^d$ ($d \geq 2$), j=0,1,. . . ,n+1, are given.
  - Find $a := t_0 < t_1 < \cdots < t_n < t_{n+1} := b$ and parametric polynomial $\boldsymbol{p} := \boldsymbol{p}_{n+1}$ of degree $\leq n+1$

$$\boldsymbol{p}(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, \ldots, n+1.$$

  - If n is large, replace $\boldsymbol{p}$ by spline $\boldsymbol{s}$.

Main problem

- If $\{t_j\}_{j=0}^{n+1}$ is known, construction of $\boldsymbol{p}$ is linear problem.
- Different choices of $\{t_j\}_{j=0}^{n+1}$ give different curves.
- How to choose the most appropriate sequence $\{t_j\}_{j=0}^{n+1}$?
- What are reasonable criteria?

Classical parameterizations

Assume $t_0 := 0$ and $t_{n+1} := 1$ (linear reparameterization).
Possible choices for $\{t_j\}_{j=1}^{n}$ are:

- Uniform:

$$t_j := \frac{j}{n+1}, \quad j = 0, 1, \ldots, n+1.$$

- Chord-length:

$$t_{j+1} = t_j + \frac{1}{L}\|\Delta \boldsymbol{T}_j\| = \frac{1}{L}\sum_{k=0}^{j}\|\Delta \boldsymbol{T}_k\|, \quad L = \sum_{k=0}^{n}\|\Delta \boldsymbol{T}_k\|.$$

- Centripetal:

$$t_{j+1} = \frac{1}{L}\sum_{k=0}^{j} \|\Delta \boldsymbol{T}_k\|^{1/2}, \quad L = \sum_{k=0}^{n} \|\Delta \boldsymbol{T}_k\|^{1/2}.$$

- Lee's generalization:

$$t_{j+1} = \frac{1}{L}\sum_{k=0}^{j} \|\Delta \boldsymbol{T}_k\|^{\mu}, \quad L = \sum_{k=0}^{n} \|\Delta \boldsymbol{T}_k\|^{\mu}, \quad \mu \in [0,1].$$

Classical parameterizations

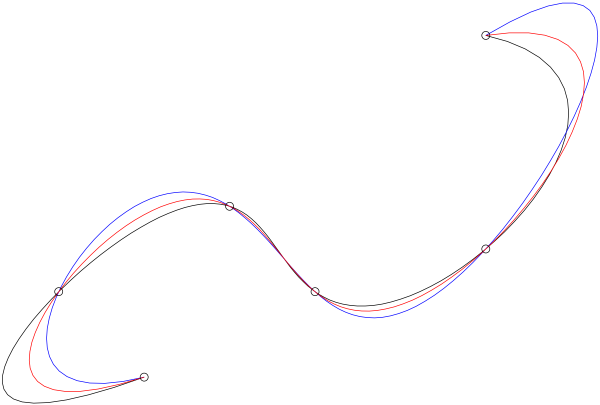

Figure: Various parameterizations: uniform (black), chord-length (blue) and centripetal (red).

Let $[t_i]\boldsymbol{T} = \boldsymbol{T}_i$ and, for $k + \ell \geq 1$,

$$
\begin{aligned}
&[t_k, t_{k+1}, \ldots, t_{k+\ell}]\boldsymbol{T} \\
&= \frac{1}{t_{k+\ell} - t_k} \left([t_{k+1}, \ldots, t_{k+\ell}]\boldsymbol{T} - [t_k, \ldots, t_{k+\ell-1}]\boldsymbol{T} \right).
\end{aligned}
$$

Interpolating polynomial $\boldsymbol{p}$ in Newton form is

$$
\boldsymbol{p}(t) = \sum_{k=0}^{n+1} (t - t_0)(t - t_1) \cdots (t - t_{k-1}) [t_0, t_1, \ldots, t_k]\boldsymbol{T}. \quad (1)
$$

A natural approach: minimize divided differences to get coefficients at higher order terms small.

- Define a nonlinear functional

$$\psi_\ell(\boldsymbol{t};\mu) := \sum_{k=0}^{n+1-\ell} \|[t_k, t_{k+1}, \ldots, t_{k+\ell}]\boldsymbol{T}\|^\mu, \qquad (2)$$

where $1 \le \ell \le n+1,\ \mu > 0$, and

$$\mathcal{D} := \{\boldsymbol{t} \in \mathbb{R}^n |\, 0 < t_1 < t_2 < \cdots < t_n < 1\}.$$

- Functional $\psi_\ell(\cdot;\mu)$ is positive on $\mathcal{D}$ and (at least for $\ell \leq 2$ unbounded at the boundary $\partial\mathcal{D}$. Namely:

Lemma

Let a sequence $\{\boldsymbol{T}_j\}_{j=0}^{n+1}$ *of distinct points be given and* $\ell = 1$ *or* $\ell = 2$*. If* $t_i \rightarrow t_j$ *for some* $i \neq j$, $0 \leq i, j \leq n+1$*, then exists at least one* $0 \leq k \leq n+1-\ell$

$$\|[t_k, t_{k+1}, \ldots, t_{k+\ell}]\boldsymbol{T}\| \rightarrow \infty.$$

As a consequence, $\psi_\ell(\cdot;\mu)$ has at least one global minimum in $\mathcal{D}$ for $\ell \leq 2$.

Particular examples

In particular, for $\ell = 1$ the global minimum is unique.

Lemma

Nonlinear functional $\psi_1(\cdot;\mu)$ *has a unique global minimum on* $\mathcal{D}$*. It is given by*

$$t_0 := 0,\ \ t_{j+1} := \frac{1}{L}\sum_{k=0}^{j} \|\Delta \boldsymbol{T}_k\|^{\frac{\mu}{\mu+1}}, \quad j = 0, 1, \ldots, n. \tag{3}$$

where

$$L = \sum_{k=0}^{n} \|\Delta \boldsymbol{T}_k\|^{\frac{\mu}{\mu+1}}.$$

For $\ell > 1$ the problem of minimization is nonlinear.
Consider only the case $n = 1$.

- Points $\boldsymbol{T}_j \in \mathbb{R}^2$, $j = 0, 1, 2$ are given.
- Find quadratic polynomial parameterization $\boldsymbol{p} : [0,1] \to \mathbb{R}^2$ such that

$$\boldsymbol{p}(t_j) = \boldsymbol{T}_j, \quad j = 0, 1, 2, \quad t_0 := 0 < t_1 < t_2 := 1.$$

- Let $t_1 := t$ and $\psi_2(t;\mu) = \|[0, t, 1]\boldsymbol{T}\|^\mu$. Find

$$\min_{0<t<1} \psi_2(t;\mu).$$

This leads to

$$g(t) = (1-t)^3\, d_0^2 - (1-t)^2\, t\, d_0\, d_1\, \cos\varphi \\ + (1-t)\, t^2\, d_0\, d_1\, \cos\varphi - t^3\, d_1^2 = 0.$$

where $d_j = \|\Delta \boldsymbol{T}_j\|$, $j = 0, 1$, and φ an angle between $\Delta \boldsymbol{T}_0$ and $\Delta \boldsymbol{T}_1$.

Lemma

Function g has precisely one root on $(0,1)$.

Particular examples

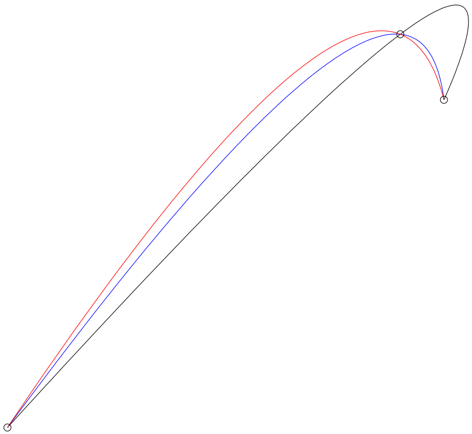

Figure: Various paramaterizations on three points: uniform (black), centripetal (blue), minimal (red).

Optimizing higher order divided differences

- For arbitrary $\ell \geq 2$ and $n > 1$ only numerical algorithms are useful.
- Uniqueness of the minimum is no more guaranteed.
- Numerical evidences: minimum is not difficult to find, best behaviour for $\ell = 2$.
- For $\ell = 2$ second order divided differences are minimized$\longrightarrow$ curvature?
- A different functional should be minimized?

Optimizing higher order divided differences

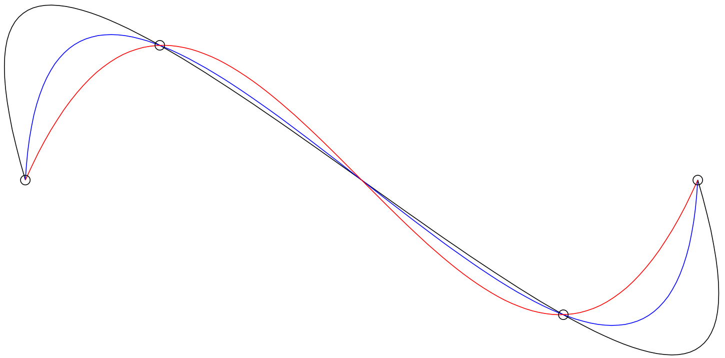

Figure: Various parameterization on four points: uniform (black), centripetal (blue), minimal with $\ell = 2$ and $\mu = 2$ (red).

Optimizing higher order divided differences

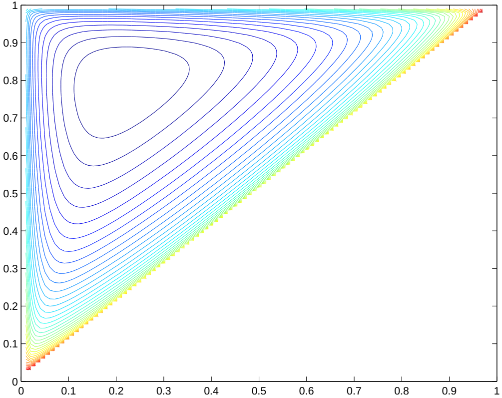

Figure: Contour plot of the functional $\psi_2(\cdot;2)$ from the previous example in logarithmic scale.

Optimizing higher order divided differences

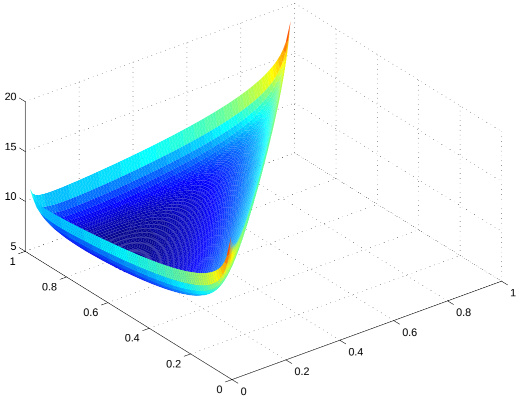

Figure: Surface plot of the functional $\psi_2(\cdot;2)$ from the previous example in logarithmic scale.

Optimizing higher order divided differences

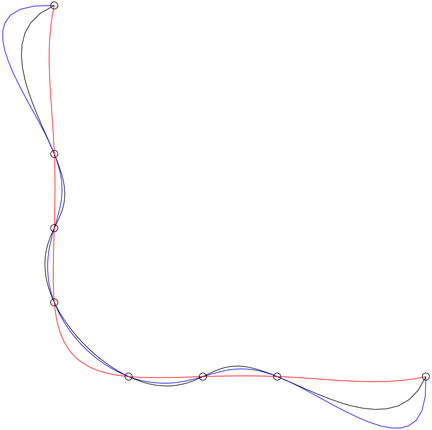

Figure: Various parameterizations on eight points: uniform (black), centripetal (blue), minimal with $\ell = 2$ and $\mu = 2$ (red).

- Suppose that a global minimum of the nonlinear functional $\psi_\ell(\cdot;\mu)$ defined by ▸ (2) is zero.
- Obviously

$$[t_k, t_{k+1}, \ldots, t_{k+\ell}]\boldsymbol{T} = \mathbf{0}, \quad k = 0, 1, \ldots, n+1-\ell,$$

- Thus

$$[t_0, t_1, \ldots, t_{k+\ell}]\boldsymbol{T} = \mathbf{0}, \quad k = 0, 1, \ldots, n+1-\ell.$$

Höllig-Koch conjecture

- By ▸ (1)

$$\mathrm{degree}(\boldsymbol{p}) \leq \ell - 1.$$

- The number of equations should be greater than the number of unknown parameters, thus

$$\ell \geq \left\lceil \frac{(d-1)n + 2\,d}{d} \right\rceil.$$

- Equality gives precisely Höllig-Koch conjecture.