

# Mean Shapes Based on Elastic Deformations

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# Motivation

In many applications a distance function between objects (shapes) is needed, e.g.

- *morphing* objects into each other in a natural way (geodesics on a shape space),
- finding a *mean shape* of some given input shapes  $S_i$   
 $i = 1, \dots, n$ ,
- *comparison* of objects (shape matching, object recognition,...).

Theory of elasticity: energy of the transformation (distance function)

## Example I: Morphing

Task: Transform first object  $S_1$  in a visual meaningful (natural) way into the second one  $S_2$ .

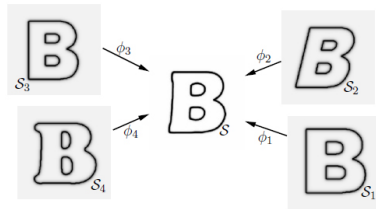


**Figure:** Morphing of two objects by minimizing an energy functional based upon elasticity theory (time discretization of a geodesic).

Source: Bar L., Sapiro G., Rumpf M., Wirth B. *Geodesics in Shape Space via Variational Time Discretization*.

## Example II: Mean Shape

Task: Find a reference shape, such that the distance (e.g. energy of the elastic deformation) of given shapes  $S_1, \dots, S_n$  to that shape is minimal.



**Figure:** Mean shape of four different input shapes obtained by minimizing an energy functional based upon elasticity theory.

Source: Rumpf M., Wirth B. *A Nonlinear Elastic Shape Averaging Approach*.

# Outline

- 1 Introduction
- 2 Shapes & Shape Spaces
  - Shapes
  - Shape Spaces
- 3 Mean Shapes
  - Minimization
  - Distance Functions
  - Relaxation

# Shapes

## Definition (Kendall [4])

The shape of an object is the geometrical information that remains after filtering out all effects of Euclidean similarity transformations, i.e. isotropic scaling, translation and rotation.

# Shapes

Let  $\emptyset \neq \Omega \subset \mathbb{R}^n$  open. Different ways to describe a *shape*:

- in general:  $\Omega$  itself or  $\partial\Omega$
- parameterizations of the boundary (embeddings modulo diffeomorphisms)
- zero level line of an appropriate scalar function  $u : \Omega \rightarrow \mathbb{R}$
- zero level set of an appropriate scalar function  $u : \Omega \rightarrow \mathbb{R}$

# Design of Shape Spaces

The invariance property of Kendall's intuitive definition of shapes can be incorporated in the design of a *shape space* with an associated metric.

Two methods:

- The shape space itself is invariant.
- The associated metric maps Euclidean motions to zero.

# Minimization Problem

Suppose a shape space  $\mathcal{A}$  of admissible shapes equipped with a given metric  $d$ . Let  $S_i \in \mathcal{A}$ ,  $i = 1, \dots, n$  be some given shapes. The *mean shape* is defined as

$$\operatorname{argmin}_{S \in \mathcal{A}} \sum_{i=1}^n d(S_i, S)^2,$$

if it exists.

# Distance Functions: Rumpf et al.

Metric based on the elastic deformation energy introduced by Rumpf et al.:

$$d(S, T)^2 = W(S, \Phi) \quad \text{constraint: } \Phi(S) = T,$$

where

$$W(S, \Phi) := \int_{\mathcal{I}(S)} w(\|D\Phi\|_2^2, \|\text{cof } D\Phi\|_2^2, \det D\Phi) dx,$$

and  $w$  denotes an appropriate kernel function.

Properties:

- invariance of  $W$  w.r.t. rigid body motions
- connection to the shape via integration domain
- nonlinear (not quadratic) energy possible

# Comparison to Linz/Infmath

Metric based on the elastic deformation energy introduced by Fuchs et al.: Let  $\gamma : [0, 1] \rightarrow \mathcal{A}$  and define

$$d(S_i, S)^2 = \inf_{\substack{\gamma(0)=S_i \\ \gamma(1)=S}} L(\gamma) = \inf_{\substack{\gamma(0)=S_i \\ \gamma(1)=S}} \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt,$$

where

$$\|\dot{\gamma}(t)\|_{\gamma(t)}^2 := \inf_{Tr_n \mathbf{u} = \dot{\gamma}(t)} E(\gamma(t), \mathbf{u}),$$

and

$$E(\gamma(t), \mathbf{u}) := \int_{\mathcal{I}(\gamma(t))} e(D\mathbf{u}) dx.$$

Here,  $e$  denotes an appropriate kernel function.

# Comparison to Linz/Infmath

Properties of the introduced metric  $d$  by Fuchs et al.:

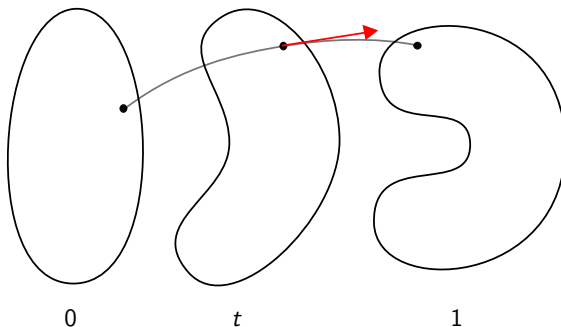
- invariance of  $e$  w.r.t. infinitesimal rigid body motions
- connection to the shape via integration domain
- linear energy (quadratic functional)
- no topological changes possible (parameterization framework)

# Comparison to Linz/Infmath

Let  $t \in [0, 1]$ .

$$\Phi(t) : \mathcal{I}(S) \rightarrow \mathbb{R}^2, \quad \gamma(t) := \Phi(t)(S)$$

$$\mathbf{u}(t) : \mathcal{I}(\gamma(t)) \rightarrow \mathbb{R}^2$$



# Comparison to Linz/Infmath

Connection between  $\Phi$  and  $\mathbf{u}$ :

$$\mathbf{u}(t)(\Phi(t)(x)) = \dot{\Phi}(t)(x) \left( = \frac{\partial}{\partial t} \Phi(t)(x) \right)$$

From  $\mathbf{u}$  to  $\Phi$ :

$$\dot{\Phi}(t)(x) = \mathbf{u}(t)(\Phi(t)(x)), \quad \Phi(0)(x) = x$$

# Comparison to Linz/Infmath

Rumpf et al.:

$$\text{minimize } W(S, \Phi(1))$$

Linz/Infmath:

$$\text{minimize } \int_0^1 E(\gamma(t), \mathbf{u}(t)) dt$$

# Hard Constraints

Minimization introduced by Rumpf et al.:

$$\operatorname{argmin}_{S \in \mathcal{A}} \sum_{i=1}^n \int_{\mathcal{I}(S_i)} w(\|D\Phi_i\|_2^2, \|\operatorname{cof} D\Phi_i\|_2^2, \det D\Phi_i) dx,$$

with constraints

$$\forall i = 1, \dots, n : \quad \Phi_i(S_i) = S.$$

# Soft Constraints

Relaxation of the constraints:  $\Phi_i(S_i) \approx S$ .

- Regularity of the mean shape by surface measure:

$$R(S) := \mathcal{H}^{d-1}(S).$$

- Match by set symmetric difference:

$$M(S_i, \Phi_i, S) := \mathcal{H}^{d-1}(S_i \Delta \Phi_i^{-1}(S)),$$

where

$$A \Delta B := (A \setminus B) \cup (B \setminus A).$$

# Relaxed Functional

Total energy by Rumpf et al.:





$$E_{\text{relaxed}}(S, \Phi_1, \dots, \Phi_n) = \frac{1}{n} \sum_{i=1}^n \left( \underbrace{W(S_i, \Phi_i)}_{\text{elastic deformation}} + \alpha \underbrace{M(S_i, \Phi_i, S)}_{\text{matching}} \right) + \beta \underbrace{R(S)}_{\text{regularity}} .$$

# Current/Future Work

Investigate the influence of integral invariants on mean shapes, whereas the main questions are:

- Which distance function (energy) should be used to compute the mean shape?
- Which distance function should be used to compute the mean of the invariants?
- Existence and uniqueness of the means?
- Does a correspondence between the input mean and the mean of the transformed objects exist?

# References

-  M. Rumpf and B. Wirth.  
A Nonlinear Elastic Shape Averaging Approach.
-  L. Bar, G. Sapiro, M. Rumpf and B. Wirth.  
Geodesics in Shape Space via Variational Time Discretization.
-  M. Fuchs, B. Jüttler, O. Scherzer and H. Yang.  
Shape Metrics Based on Elastic Deformations.
-  D.G. Kendall.  
*The diffusion of shape.*  
Advances in Applied Probability 9, 428–430 (1977)

# Appendix: Existence

A minimizer

$$\Phi_j \in W^{1,p}(\Omega, \mathbb{R}^d) \cap \{\Phi : \Omega \rightarrow \Omega \mid \Phi|_{\partial\Omega}(x) = x, \det(D\Phi) > 0\}$$

and

$$v \in W^{1,2}(\Omega)$$

of  $E_{\text{relaxed}}^\epsilon$  exists provided that

- all regularization parameters are positive,
- $w$  satisfies a coercivity condition.