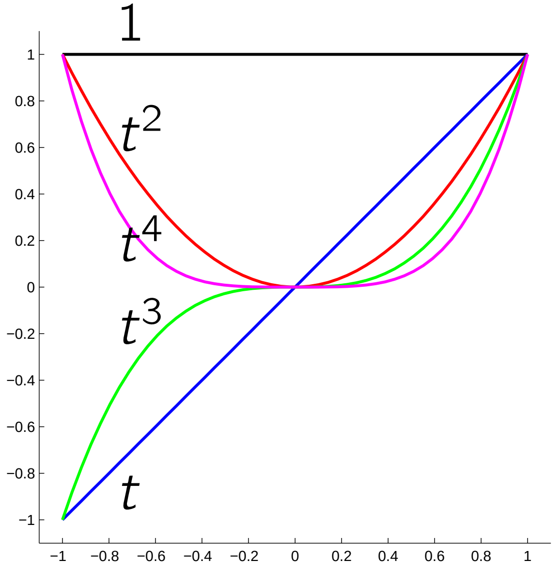
Curves for CAGD

FSP–Tutorial

Polynomial Curves

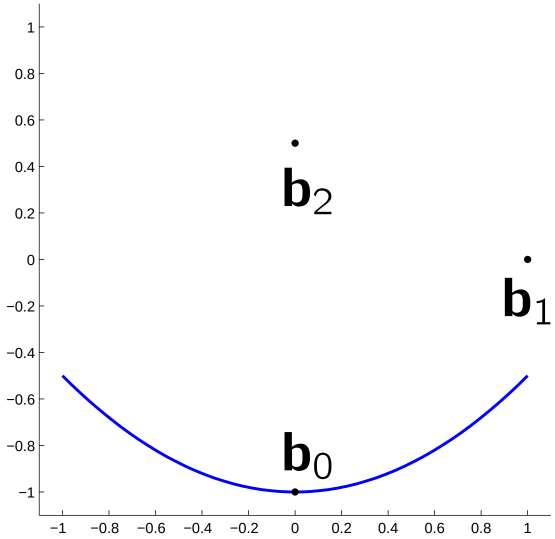
A segment of a **polynomial curve** of degree n in $\mathbb{R}^d$ $(d = 2, 3)$ is parametrized with respect to the *monomial basis* $t^0, t^1, \ldots, t^n$ by

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i t^i, \;\; t \in [a, b].$$



The coefficient vectors $\mathbf{b}_i$ do not have a geometric interpretation.

Example: Parabola $\mathbf{b}(t) = \mathbf{b}_0 t^0 + \mathbf{b}_1 t^1 + \mathbf{b}_2 t^2$



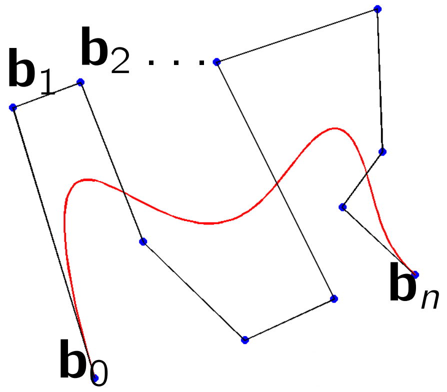
Bézier Curves and Bernstein Basis

Given $n+1$ points $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_n$ in space $\mathbb{R}^d$, the *Bézier curve* defined by these *control points* is

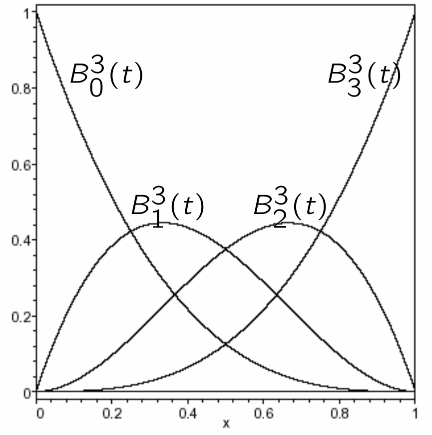
$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t), \;\; t \in [0,1]. \qquad (1)$$

$\mathbf{b}(t)$ is a polynomial curve parametrized in the *Bernstein basis*

$$B_i^n(t) = \binom{n}{i} (1-t)^{n-i} t^i. \qquad (2)$$



Control polygon and Bézier curve

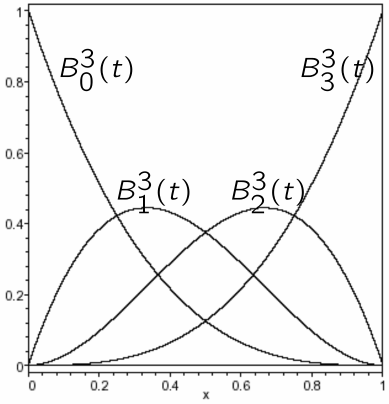


Cubic Bernstein polynomials.

Geometric Properties of the Bernstein Basis

$t \in [0,1]$

- Recursion: $B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t), \quad i = 0, \dots, n.$
- Derivatives: $\frac{d}{dt}B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right).$



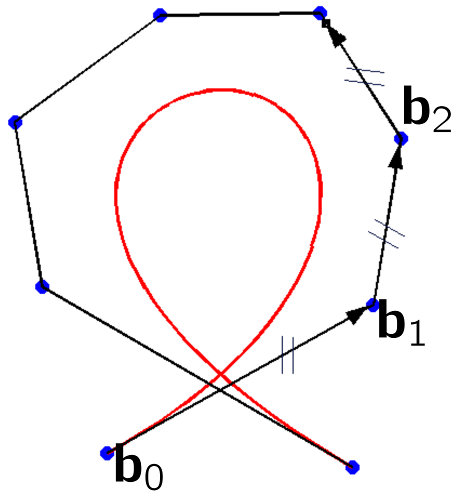
- Non-negativity: $B_i^n(t) \geq 0$.
- Partition of unity: $\sum_{i=0}^n B_i^n(t) = 1$.
- Symmetry: $B_i^n(t) = B_{n-i}^n(1-t)$.

- $B_i^n(t)$ has a i-fold zero at $t = 0$, and a $(n-i)$-fold zero at $t = 1$.
- $B_i^n(t)$ has exactly one maximum in $I := [0,1]$ at $t = \frac{i}{n}$.

Derivatives of Bézier Curves

Derivative $\dot{\mathbf{b}}(t)$ of a Bézier curve:

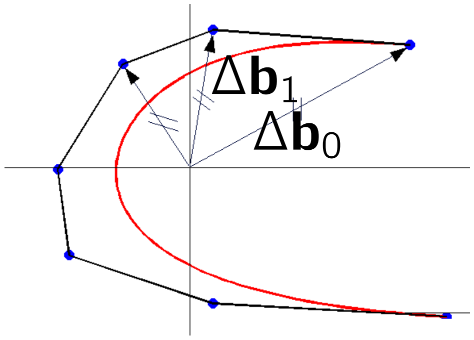
$$\begin{aligned}\dot{\mathbf{b}}(t) &= \sum_{j=0}^{n-1} B_j^{n-1}(t)(\mathbf{b}_{j+1} - \mathbf{b}_j) \\ &= n\sum_{j=0}^{n-1} B_j^{n-1}(t)\triangle\mathbf{b}_j.\end{aligned}$$



Bézier curve

Tangent vectors at $t = 0$ and $t = 1$:

$$\begin{aligned}\dot{\mathbf{b}}(0) &= n(\mathbf{b}_1 - \mathbf{b}_0), \\ \dot{\mathbf{b}}(1) &= n(\mathbf{b}_n - \mathbf{b}_{n-1}),\end{aligned}$$



Hodograph

Higher order derivatives can be computed in a similar way.

Moving Control Points

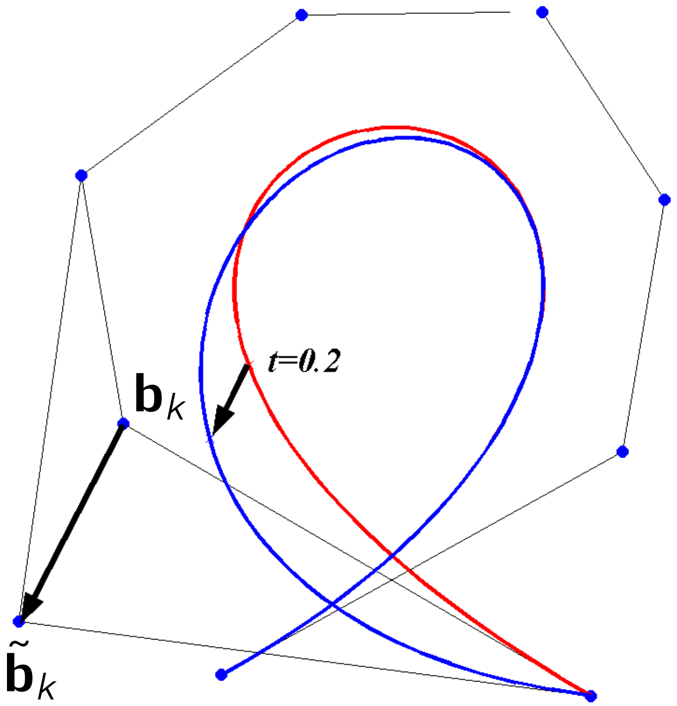
Control point $\mathbf{b}_k$ is moved to

$$\tilde{\mathbf{b}}_k = \mathbf{b}_k + \mathbf{v}.$$

The curve changes to

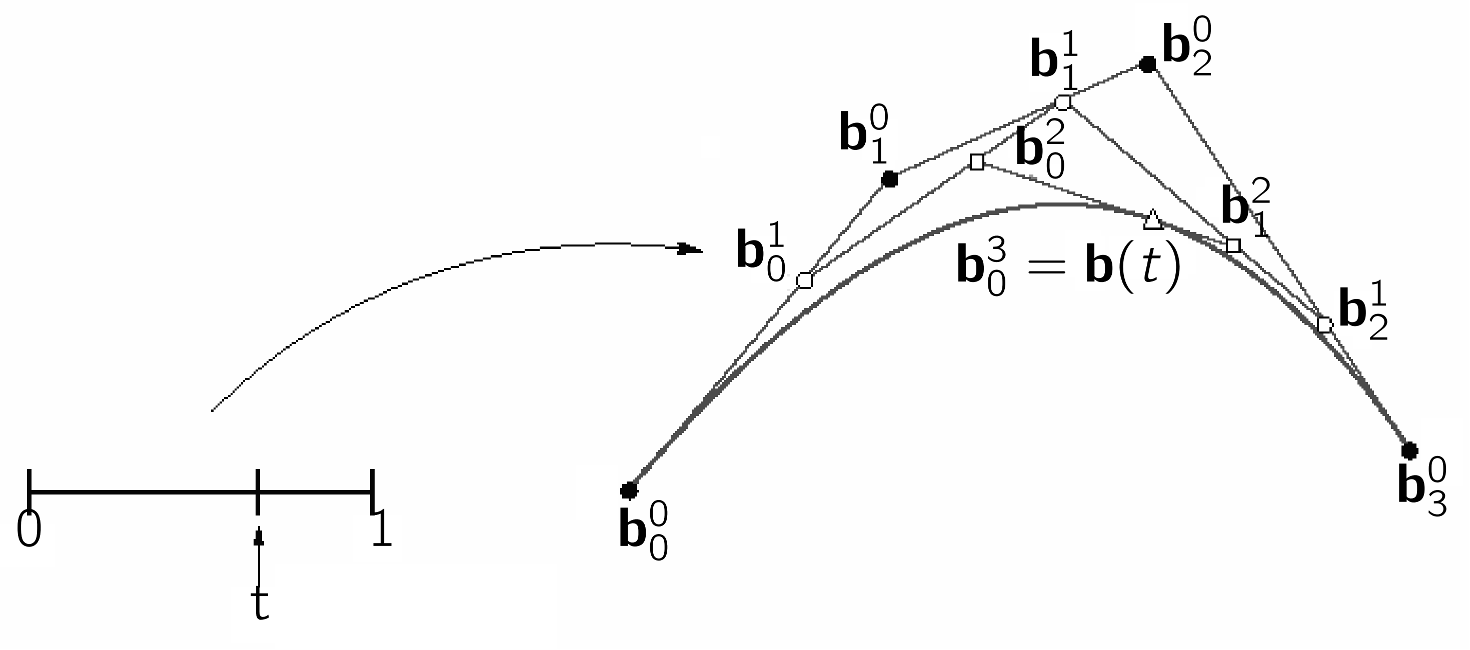
$$\begin{aligned} \tilde{\mathbf{b}}(t) &= \sum_{i=0}^{n} B_i^n(t)\mathbf{b}_i + B_k^n(t)\mathbf{v}, \\ &= \mathbf{b}(t) + B_k^n(t)\mathbf{v}. \end{aligned}$$

The shape of the curve changes globally.



Algorithm of de Casteljau

- Input: control polygon $\mathbf{b}_0, \ldots, \mathbf{b}_n$ and $t \in [0,1]$
- Initialize: $\mathbf{b}_i^0 := \mathbf{b}_i,\ i = 0, \ldots, n.$
- Recursion:
  $\mathbf{b}_i^r = (1-t)\mathbf{b}_i^{r-1} + t\mathbf{b}_{i+1}^{r-1}, r = 1, \ldots, n, i = 0, \ldots, n-r$
- Result: $\mathbf{b}_0^n = \mathbf{b}(t)$



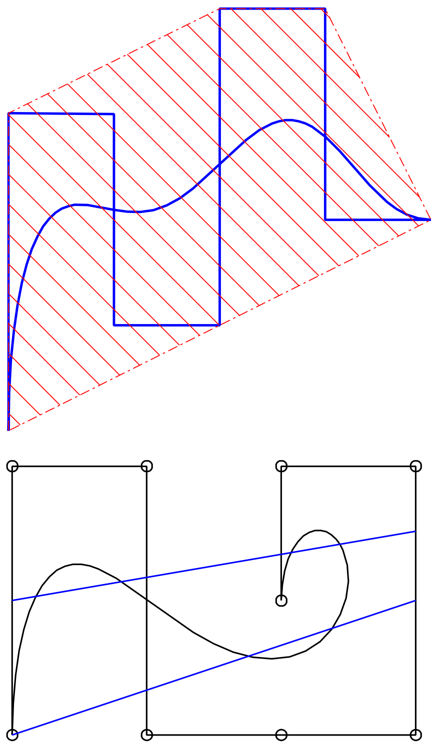
Some Properties of Bézier Curves

- $\mathbf{b}(t) = \sum_i B_i^n(t)\mathbf{b}_i$ is a polynomial curve of degree n.
- End point interpolation: $\mathbf{b}(0) = \mathbf{b}_0, \mathbf{b}(1) = \mathbf{b}_n$
- Affine invariance: Constructing the curve from the image points $\mathbf{b}_i'$ is equivalent to applying an affine mapping $\alpha : \mathbf{x}' = A \cdot \mathbf{x} + \mathbf{a}$ to the curve $\mathbf{b}(t)$.

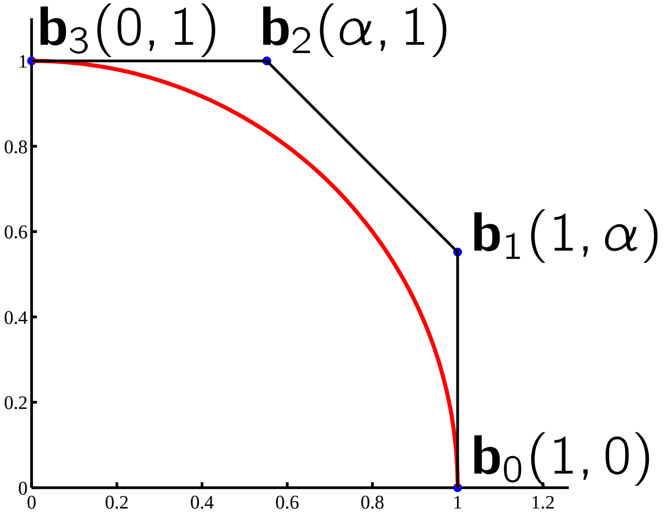
$$\begin{aligned}
\tilde{\mathbf{b}}(t) &= \sum B_i^n(t)\mathbf{b}_i' = \sum B_i^n(t)(A \cdot \mathbf{b}_i + \mathbf{a}) \\
&= A \cdot \left(\sum B_i^n(t)\mathbf{b}_i\right) + \sum B_i^n(t)\mathbf{a} = A \cdot \mathbf{b}(t) + \mathbf{a} = \mathbf{b}(t)'.
\end{aligned}$$

Some Properties of Bézier Curves 2

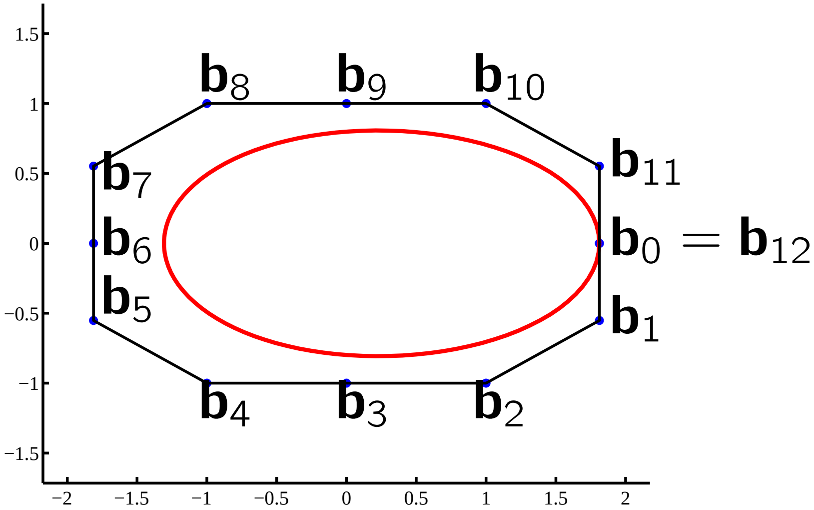
- Convex hull property: The Bézier curve $\mathbf{b}(t)$ lies completely in the convex hull of the control points $\mathbf{b}_i$, $i = 0, \ldots, n$.
- Variation diminishing property: The number of the intersections of $\mathbf{b}(t)$ with a hyperplane H ($d = 2$: line; $d = 3$: plane) is not greater than the number of the intersections of the control polygon $\mathbf{b}_0, \ldots, \mathbf{b}_n$ with H.
- A *convex* control polygon implies a *convex* Bézier curve.



Examples



Approximating Bézier curve for the quarter of a circle



Approximating Bézier curve for an ellipse

Polar Form

Generalized algorithm of de Casteljau

$$\mathbf{b}_i^r(t_1,\dots,t_r) := (1-t_r)\mathbf{b}_i^{r-1}(t_1,\dots,t_{r-1}) + t_r\mathbf{b}_{i+1}^{r-1}(t_1,\dots,t_{r-1}),$$

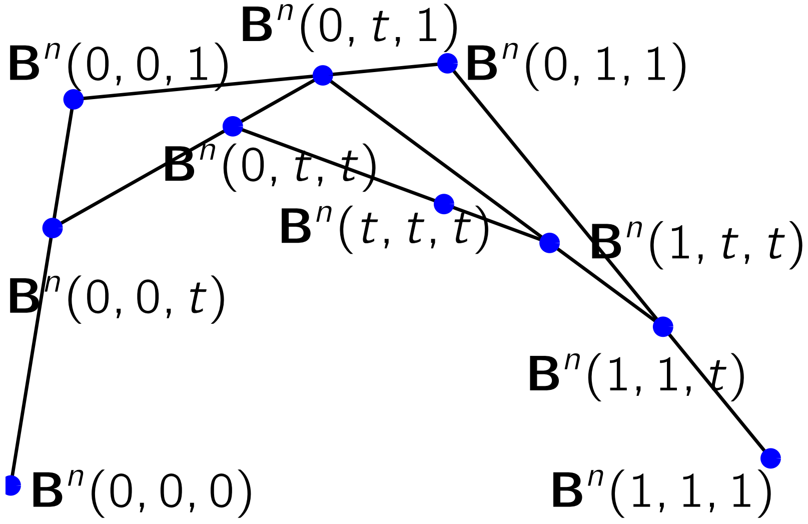
for $r = 1,\dots,n$, $i = 0,\dots,n-r$.

$\mathbf{B}^n : \mathbb{R}^n \to \mathbb{R}^d$

$(t_1,\dots,t_n) \mapsto \mathbf{B}^n(t_1,\dots,t_n)$

$:= \mathbf{b}_0^n(t_1,\dots,t_n)$.

- $\mathbf{B}^n$ is *symmetric*.
- $\mathbf{B}^n$ is *multiaffine*.



Properties of the Polar Form

- $\mathbf{B}^n$ is *symmetric*: After the first $i-1$ steps in the de Casteljau scheme we have the polygon $\mathbf{b}_0^{i-1}, \ldots, \mathbf{b}_{n-i+1}^{i-1}$. Now we find:

$$\begin{aligned}
\mathbf{b}_j^i &= (1-t_i)\mathbf{b}_j^{i-1} + t_i\mathbf{b}_{j+1}^{i-1}, \\
\mathbf{b}_j^{i+1} &= (1-t_{i+1})\left[(1-t_i)\mathbf{b}_j^{i-1} + t_i\mathbf{b}_{j+1}^{i-1}\right] + t_{i+1}\left[(1-t_i)\mathbf{b}_{j+1}^{i-1} + t_i\mathbf{b}_{j+2}^{i-1}\right] \\
&= (1-t_i)(1-t_{i+1})\mathbf{b}_j^{i-1} + \left[(1-t_{i+1})t_i + (1-t_i)t_{i+1}\right]\mathbf{b}_{j+1}^{i-1} + t_i t_{i+1}\mathbf{b}_{j+2}^{i-1}.
\end{aligned}$$

- $\mathbf{B}^n$ is *multiaffine*:

$$\mathbf{B}^n(t_1, \ldots, t_{i-1}, (1-\alpha)a_1 + \alpha a_2, t_{i+1}, \ldots, t_n) =$$

$$= (1-\alpha)\mathbf{B}^n(t_1, \ldots, t_{i-1}, a_1, t_{i+1}, \ldots, t_n) + \alpha\mathbf{B}^n(t_1, \ldots, t_{i-1}, a_2, t_{i+1}, \ldots, t_n).$$

Elementary Symmetric Functions

$$S_i(t_1, \ldots, t_n) = \sum_{1 \le j_1 < j_2 < \ldots < j_i \le n} t_{j_1} \cdots t_{j_i}, \quad i = 0, \ldots, n.$$

$$\begin{aligned} n = 3: \quad S_0(t_1, t_2, t_3) &= 1, \\ S_1(t_1, t_2, t_3) &= t_1 + t_2 + t_3, \\ S_2(t_1, t_2, t_3) &= t_1 t_2 + t_1 t_3 + t_2 t_3, \\ S_3(t_1, t_2, t_3) &= t_1 t_2 t_3. \end{aligned}$$

- Any linear combination $\sum_{i=0}^{n} S_i(t_1, \ldots, t_n)\mathbf{c}_i, \mathbf{c}_i \in \mathbb{R}^d$ is a symmetric multiaffine map $\mathbb{R}^n \to \mathbb{R}^d$.
- A map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^d$ is multiaffine and symmetric exactly if it is a linear combination $\mathbf{F} = \sum_{i=0}^{n} S_i(t_1, \ldots, t_n)\mathbf{c}_i$ of elementary symmetric functions.

Polar Form of a Bézier Curve

To each polynomial curve $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ of degree n there exists exactly one symmetric multiaffine map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^d$ with $\mathbf{F}(u, \ldots, u) = \mathbf{f}(u)$. $\mathbf{F}$ is called **polar form** of $\mathbf{f}$.

Example: The cubic polynomial curve

$$\mathbf{f}(u) = \mathbf{a}_0 + u\mathbf{a}_1 + u^2\mathbf{a}_2 + u^3\mathbf{a}_3$$

possesses the polar form

$$\mathbf{F}(u_1, u_2, u_3) = \mathbf{a}_0 + \frac{1}{3}(u_1 + u_2 + u_3)\mathbf{a}_1 + \frac{1}{3}(u_1 u_2 + u_1 u_3 + u_2 u_3)\mathbf{a}_2 + u_1 u_2 u_3 \mathbf{a}_3.$$

Polar Form of a Bézier Curve 2

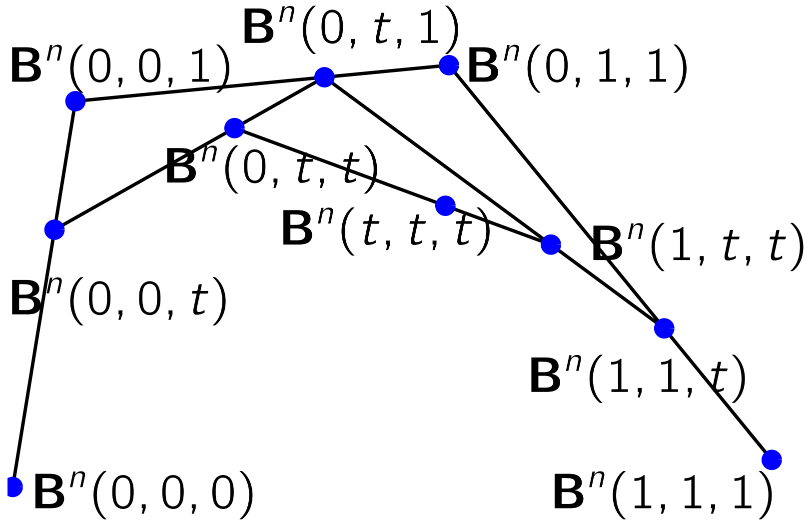
To each Bézier curve (polynomial curve) $\mathbf{b}^n \subset \mathbb{R}^d$ there exists a unique polar form $\mathbf{B}^n : \mathbb{R}^n \to \mathbb{R}^d$ with

$$\mathbf{B}^n(t, \ldots, t) = \mathbf{b}^n(t).$$

The Bézier curve segment to the parameter interval $[0, 1]$ possesses the Bézier points $\mathbf{b}_i = \mathbf{B}^n\,(\underbrace{0, \ldots, 0}_{n-i}, \underbrace{1, \ldots, 1}_{i})$.

Further, we see that

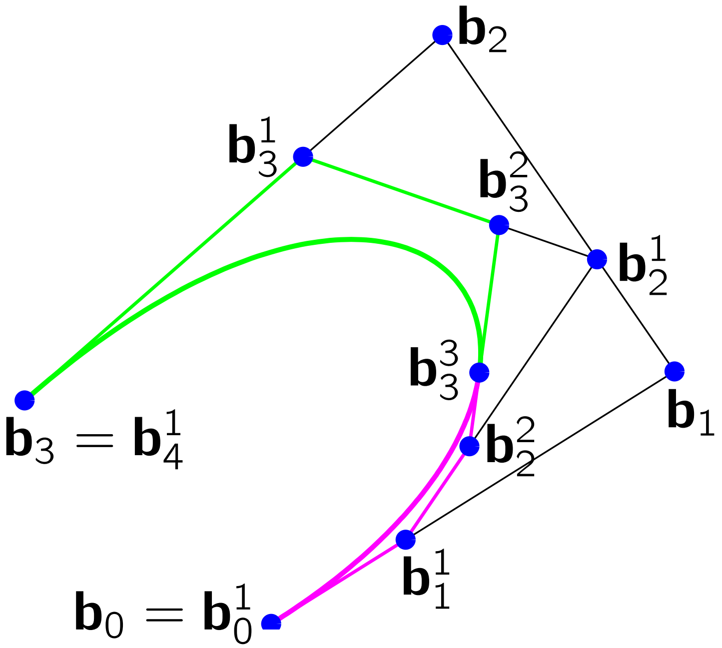
$$\begin{aligned} \mathbf{b}_0^i &= \mathbf{B}^n(\underbrace{0, \ldots, 0}_{n-i}, \underbrace{t, \ldots, t}_{i}), \\ \mathbf{b}_i^{n-i} &= \mathbf{B}^n(\underbrace{t, \ldots, t}_{n-i}, \underbrace{1, \ldots, 1}_{i}). \end{aligned}$$



Subdivision of Bézier Curves

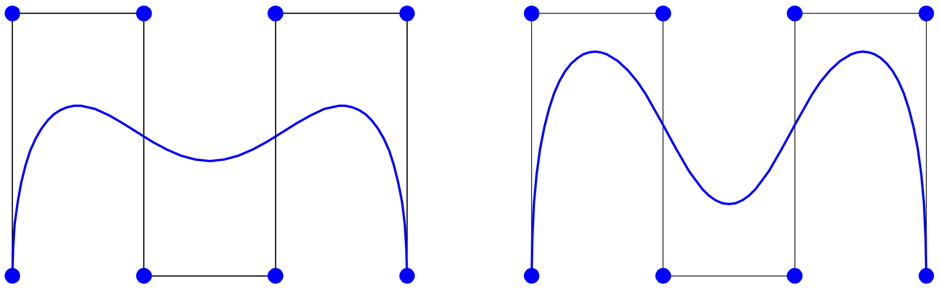
Bézier curve $\mathbf{b}^n(u)$ with control points $(\mathbf{b}_0, \ldots, \mathbf{b}_n)$. The de Casteljau algorithm yields the control points $(\mathbf{c}_0, \ldots, \mathbf{c}_n)$ and $(\mathbf{d}_0, \ldots, \mathbf{d}_n)$ of $\mathbf{b}^n(u)$ over the subintervals $[0, u]$ and $[u, 1]$, respectively. We have

$$\mathbf{c}_i = \mathbf{b}_0^i(u), \quad \mathbf{d}_i = \mathbf{b}_i^{n-i}(u), \; i = 0, \ldots, n.$$

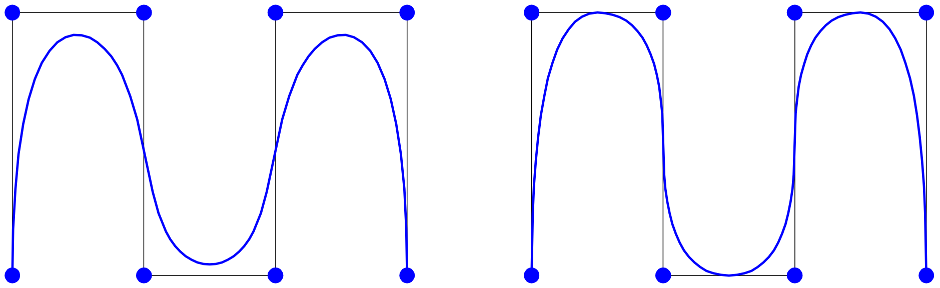


Repeated subdivision yields a sequence of polygons that converges to the curve. The refinement is a *corner cutting*.

B–Spline Curves – Introduction



degree 7 (Bézier) degree 5

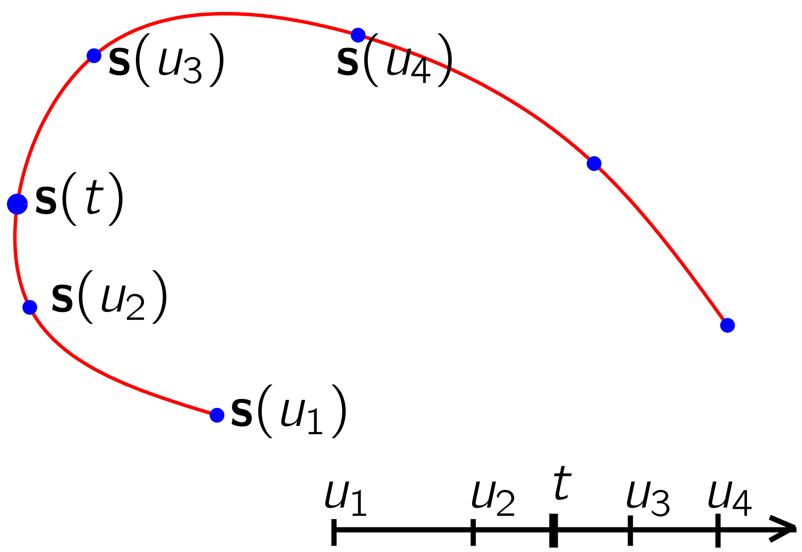


degree 3 degree 2

Control polygons and B–Spline curves of different degrees

Bézier curves have fixed degree depending on the number of control points. B–Spline curves are composed of Bézier curve segments.

B–Spline Curves



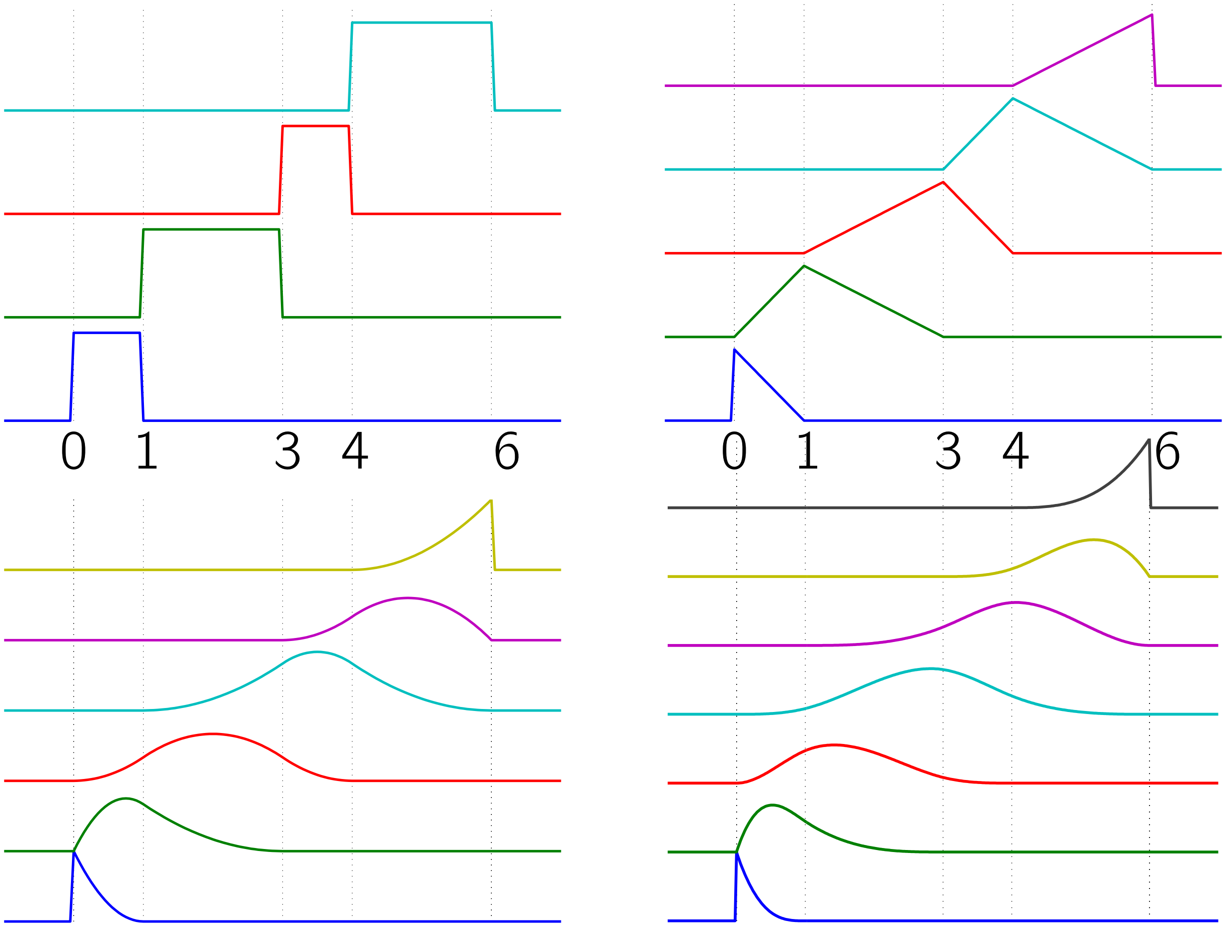
A B–Spline curve

$$\mathbf{s}(u) = \sum_{i=0}^{m} N_i^n(u)\mathbf{d}_i$$

is determined by the control points $\mathbf{d}_i$ and the degree of the basis functions $N_i^n(u)$.

regular case: # control points = degree + # segments

B–Spline Basis Functions



Constant and linear B–Splines for the knots 0, 1, 3, 4, 6 and 0, 0, 1, 3, 4, 6, 6.

Quadratic and cubic B–Splines for the knots 0, 0, 0, 1, 3, 4, 6, 6, 6 and $0^{(4)}, 1, 3, 4, 6^{(4)}$.

Properties of B–Splines

$$N_i^0(u) \quad := \quad \begin{cases} 1 & \text{for} \quad t_i \le u < t_{i+1} \\ 0 & \text{else} \end{cases}$$

$$N_i^r(u) \quad := \quad \frac{u - t_i}{t_{i+r} - t_i} N_i^{r-1}(u) + \frac{t_{i+r+1} - u}{t_{i+r+1} - t_{i+1}} N_{i+1}^{r-1}(u), \quad 1 \le r \le n.$$

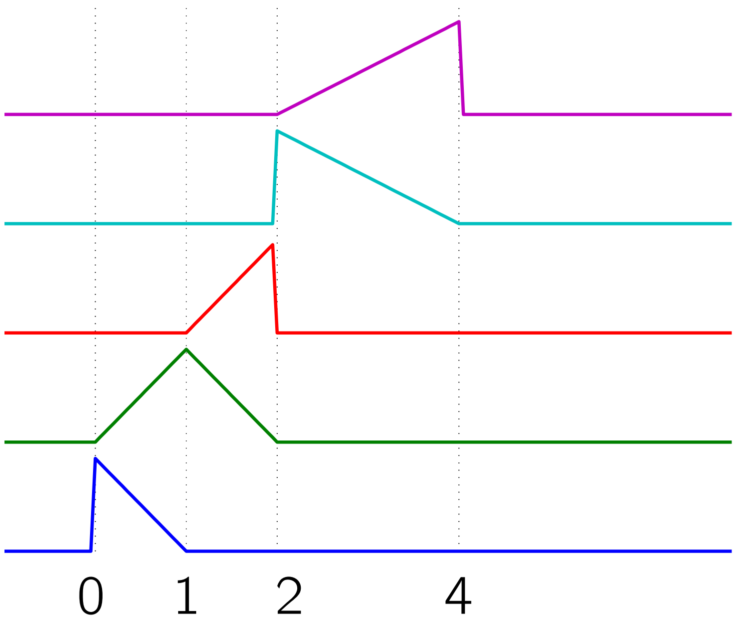
- $N_i^p(u)$ is a function composed of polynomials.
- The restriction of $N_i^p(u)$ to (t_j, t_{j+1}) is a polynomial of degree p.
- Nonnegativity:

$$N_i^p(u) > 0 \text{ for } u \in (t_i, t_{i+n+1}),\ N_i^p(u) = 0 \text{ for } u \notin [t_i, t_{i+n+1}].$$

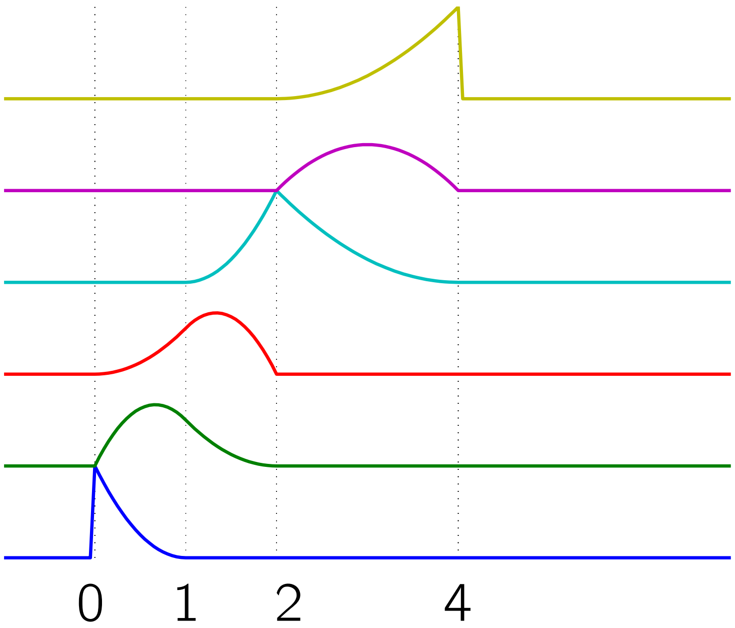
- The interval $[t_i, t_{i+p+1}]$ is called *support* of $N_i^p(u)$.

Properties of B–Splines 2

- Partition of unity: $\sum_{j=i-p}^{i} N_j^p(u) = 1$.
- $N_i^p(u)$ is C^{p-k}-continuous at a knot of multiplicity k.



knot vector $[0, 0, 1, 2, 2, 4, 4]$



knot vector$[0, 0, 0, 1, 2, 2, 4, 4, 4]$

- On any interval $[t_i, t_{i+1})$ at most $p + 1$ basis functions of degree p are non-zero, $N_{i-p}^p(u), N_{i-p+1}^p(u), \ldots,$ and $N_i^p(u)$.

Polar Form of B–Splines

The restriction of $N_i^n(u)$ to (t_j, t_{j+1}) is a polynomial $N_{i,j}^n(u)$.

$\Rightarrow$ associated *polar form* $P_{i,j}^n(u_1, \ldots, u_n)$.

The polar form $P_{i,j}^n$ of $N_{i,j}^n$ is given by

$$P_{i,j}^0() = \delta_{i,j},$$

$$P_{i,j}^r(u_1, \ldots, u_r) = \tfrac{u_r - t_i}{t_{i+r} - t_i} P_{i,j}^{r-1}(u_1, \ldots, u_{r-1}) + \tfrac{t_{i+r+1} - u_r}{t_{i+r+1} - t_{i+1}} P_{i+1,j}^{r-1}(u_1, \ldots, u_{r-1}).$$

$P_{i,j}^r$ is symmetric and multiaffine and $P_{ij}^r(u, \ldots, u) = N_{ij}(u)$.

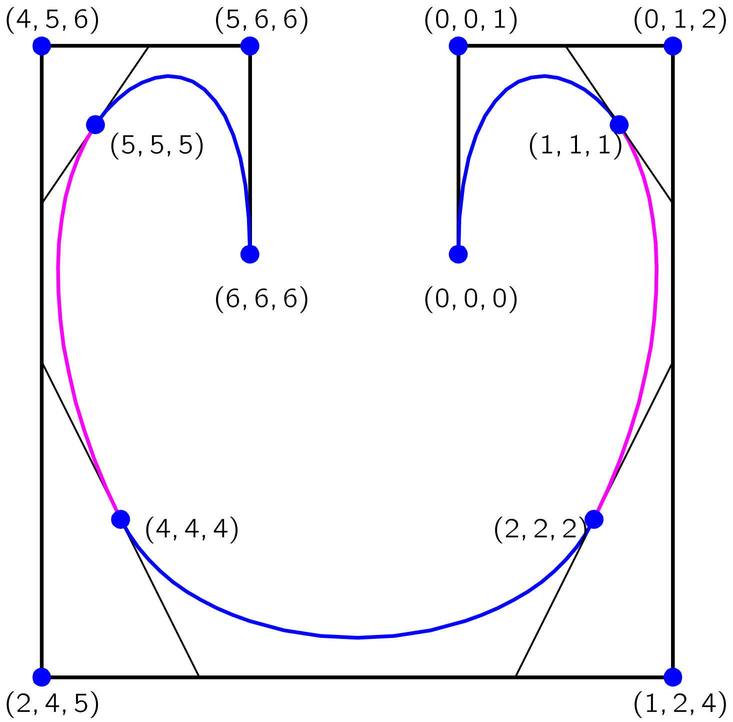
Main theorem on B–Spline curves

B–Spline curve

$$\mathbf{s}(u) = \sum_{i=0}^{m} N_i^n(u)\mathbf{d}_i,\ \mathbf{d}_i \in \mathbb{R}^d,$$

knots $T = (t_0, \ldots, t_{m+n+1})$. Polar form $\mathbf{S}_j(u_1, \ldots, u_n)$ of the restriction of $\mathbf{s}$ onto (t_j, t_{j+1}).
The control points $\mathbf{d}_l$ (for $l \le j \le l+n$) can be expressed as

$$\mathbf{d}_l = \mathbf{S}_j(t_{l+1}, \ldots, t_{l+n}).$$



$T = (0^4, 1, 2, 4, 5, 6^4)$

The normalized B-Splines $\{N_i^n | i = 0, \ldots, m\}$ are *linearly independent*.

Geometric Properties of B–Spline Curves

De Boor's algorithm: $\mathbf{d}_i^0 := \mathbf{d}_i$,

$$\mathbf{d}_i^r := (1 - \frac{u - t_i}{t_{i+n+1-r} - t_i})\mathbf{d}_{i-1}^{r-1} + \frac{u - t_i}{t_{i+n+1-r} - t_i}\mathbf{d}_i^{r-1} \text{ for } l - n + r \leq i \leq l.$$

$\Longrightarrow$ Curve point $\mathbf{s}(u) = \mathbf{d}_l^n$.

- *Affine invariance*: true because of $\sum_{i=0}^{m} N_i^n(u) = 1$.
- *Convex hull property*.
- *Subdivision property*.
- *Local control*: Changing a control point $\mathbf{d}_l$ influences at most $n + 1$ segments ((t_l, t_{l+n+1})).
- In each recursion depth r ($r \geq 1$) the knot u is inserted as new knot. At the end, u is inserted n–fold.

Knot Insertion

B–spline curve $\mathbf{s}(u) = \sum_{i=0}^{m} N_i^n(u)\mathbf{d}_i$ over $T = (t_0, \dots, t_{m+n+1})$
Insertion of a new knot t at $t_l \le t < t_{l+1}$ yields

$$\mathbf{s}(u) = \sum_{i=0}^{m+1} N_i^{n*}(u)\mathbf{d}_i^*$$

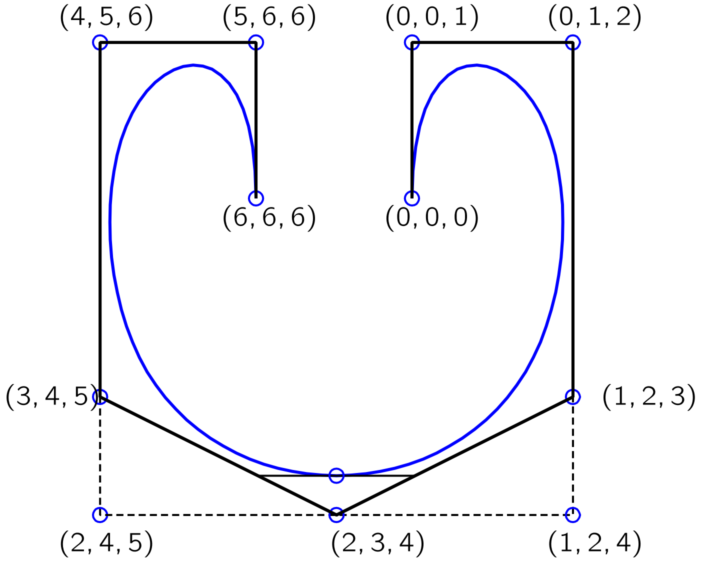
over the new knot vector $T^* = (t_0, \dots, t_l, t, t_{l+1}, \dots, t_{m+n+1})$.
With

$$\begin{aligned} t &= (1 - \tfrac{t - t_i}{t_{i+n} - t_i})t_i + \tfrac{t - t_i}{t_{i+n} - t_i} t_{i+n} \\ &= (1 - a_i)t_i + a_i t_{i+n} \end{aligned}$$

we obtain

$$\mathbf{d}_i^* = (1 - a_i)\mathbf{d}_{i-1} + a_i\mathbf{d}_i$$

with $a_i = 1$ if $i \le l - n$, $a_i = 0$ if $l + 1 \le i$,
else $a_i = t - t_i / t_{i+n} - t_i$.



$T = (0^4, 1, 2, 3, 4, 5, 6^4)$

B–Spline Curve Subdivision

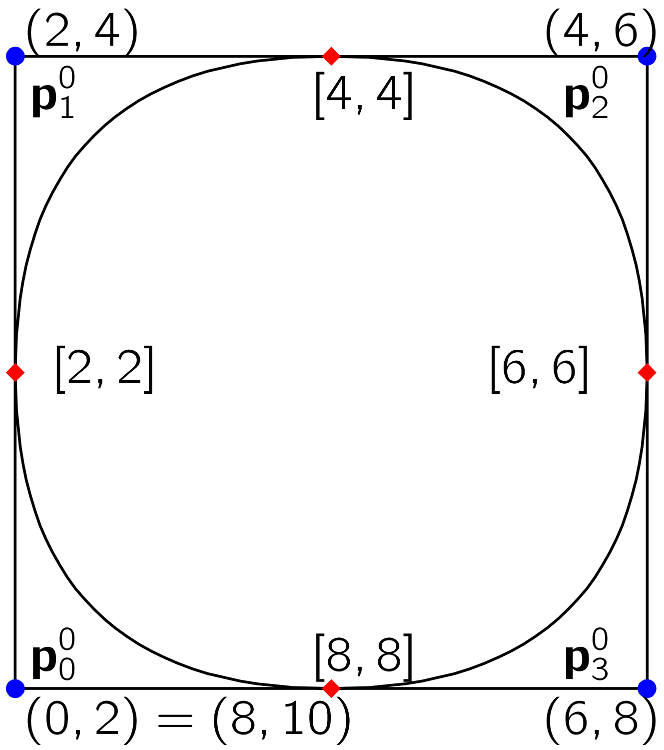
Given a control polygon P of a closed B–Spline curve.

- Splitting: introduce midpoints of segments.
- Averaging: Replace new segments by their centers.

Subdivision: *Splitting* $+ (k-1)$-times *Averaging* results in a refined control polygon of a B–Spline curve of degree k.

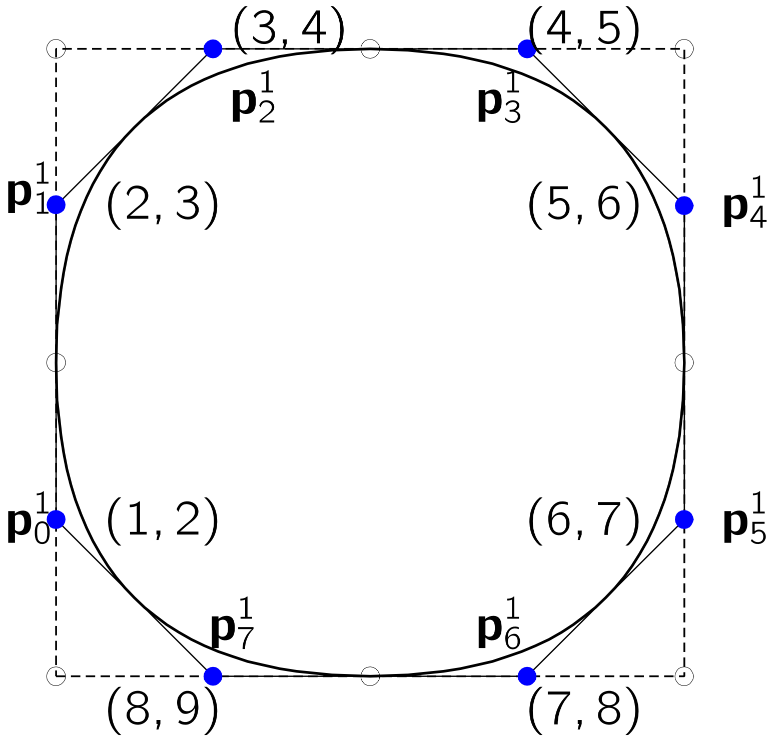
Quadratic B–Spline Subdivision – Chaikin

Splitting



$T = (\ldots, 0, 2, 4, 6, 8, \ldots)$

Averaging

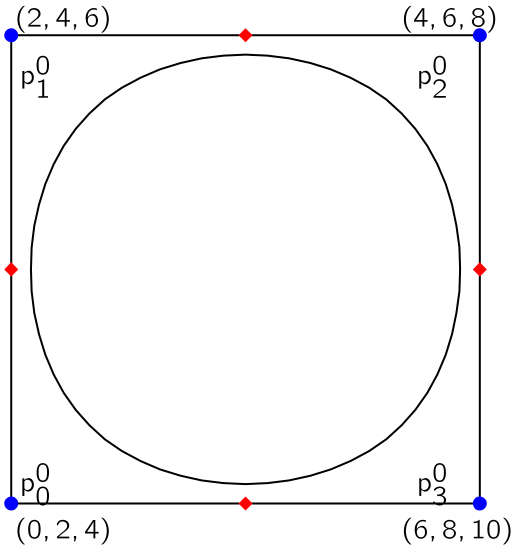


$T = (\ldots, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots)$

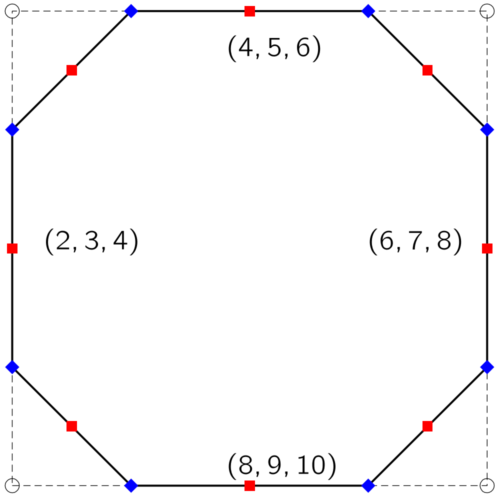
$$
\begin{aligned}
\mathbf{p}_{2i}^{k+1} &= \tfrac{3}{4}\mathbf{p}_i^k + \tfrac{1}{4}\mathbf{p}_{i+1}^k \\
\mathbf{p}_{2i+1}^{k+1} &= \tfrac{1}{4}\mathbf{p}_i^k + \tfrac{3}{4}\mathbf{p}_{i+1}^k
\end{aligned}
$$

Cubic B–Spline Subdivision

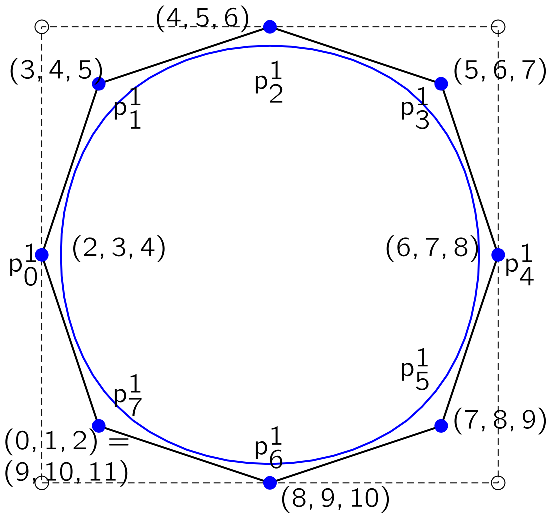
Splitting



Averaging



Averaging



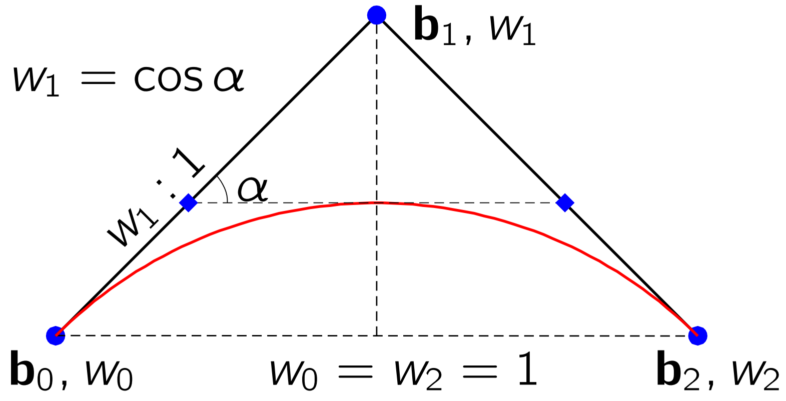
$T = (\ldots, 0, 2, 4, 6, 8, 10, \ldots)$ $\quad$ $T = (\ldots, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots)$

$$\begin{aligned} \mathbf{p}_{2i}^{k+1} &= \tfrac{4}{8}\mathbf{p}_i^k + \tfrac{4}{8}\mathbf{p}_{i+1}^k \\ \mathbf{p}_{2i+1}^{k+1} &= \tfrac{1}{8}\mathbf{p}_i^k + \tfrac{6}{8}\mathbf{p}_{i+1}^k + \tfrac{1}{8}\mathbf{p}_{i+2}^k \end{aligned}$$

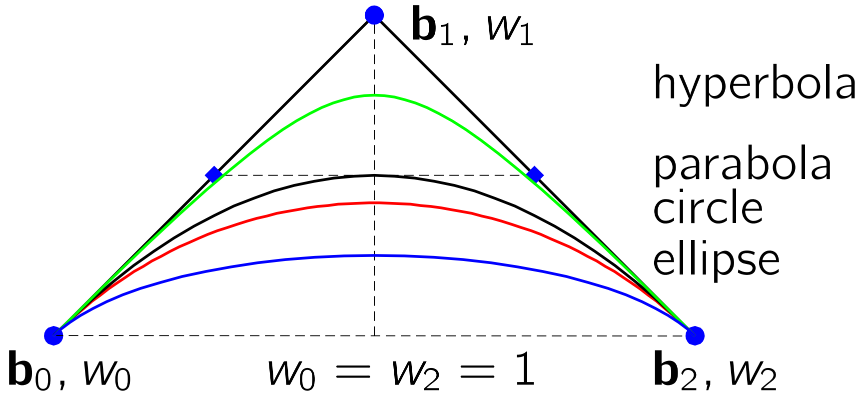
Rational Bézier Curves

Circles, ellipses, hyperbolas and piecewise rational curves cannot be represented exactly as (piecewise) polynomial curves. We introduce *weights* w_i and rational parametrizations

$$\mathbf{b}(t) = \frac{1}{\sum_i^n B_i^n(t) w_i} \sum_i^n B_i^n(t) \mathbf{b}_i w_i ,$$

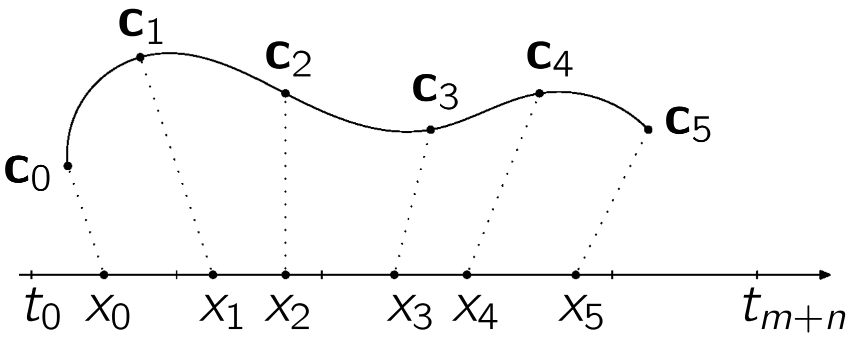


Circle



Interpolation and Approximation

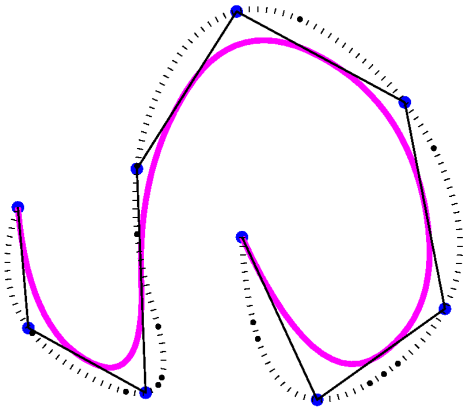
- Given:
  - M data points $\mathbf{c}_0, \ldots, \mathbf{c}_{M-1}$ with M corresponding parameter values $x_0, \ldots, x_{M-1}$
  - m–dimensional space with basis functions $N_0^n, \ldots, N_{m-1}^n$ over a knot vector T
- Find: Curve $\mathbf{s}(u) = \sum_{i=0}^{m-1} N_i(u)^n \mathbf{d}_i$ that interpolates / approximates the data points.



The given points $\mathbf{c}_0, \ldots, \mathbf{c}_{M-1}$ are to be interpolated / approximated at given parameter values $x_0, \ldots, x_{M-1}$

Interpolation and Approximation - Cases

- $M = m$: number of conditions = number of degree of freedom → interpolation
- $M > m$: overdetermined system of equations → approximation
- $M < m$: underdetermined system of equations → "constrained" interpolation / approximation (additional constraints)



Approximating and interpolating B-spline curve

B–Spline Interpolation

- Given:
  - m data points $\mathbf{c}_0, \ldots, \mathbf{c}_{m-1}$ with m corresponding parameter values $x_0, \ldots, x_{m-1}$
  - m–dimensional space with basis functions $N_0^n, \ldots, N_{m-1}^n$ over a knot vector $T = (t_0 \leq \ldots \leq t_{m+n})$
- Find: Control points $\mathbf{d}_0, \ldots, \mathbf{d}_{m-1}$ such that $\mathbf{s}(u) = \sum_{i=0}^{m-1} N_i^n(u)\mathbf{d}_i$ interpolates the data points $\mathbf{c}_0, \ldots, \mathbf{c}_{m-1}$ at parameter values $x_0, \ldots, x_{m-1}$: $\mathbf{s}(x_i) = \mathbf{c}_i \quad \forall i$

$$\begin{pmatrix} N_0^n(x_0) & \ldots & N_{m-1}^n(x_0) \\ \vdots & & \vdots \\ N_0^n(x_{m-1}) & \ldots & N_{m-1}^n(x_{m-1}) \end{pmatrix} \begin{pmatrix} \mathbf{d}^0 \\ \vdots \\ \mathbf{d}^n \end{pmatrix} = \begin{pmatrix} \mathbf{c}^0 \\ \vdots \\ \mathbf{c}^n \end{pmatrix}$$

- $A = (N_i^n)_{ij}$ has $n+1$ non-zero entries per row (support of $N_i^n : [t_i, t_i + n + 1]$)

Special Case: Cubic Spline Interpolation

- Given:
  - Degree $n = 3$
  - $m - 2$ data points $\mathbf{c}_0, ...\mathbf{c}_{m-3}$
  - m-dimensional space of basis functions $N_0^3, \ldots, N_{m-1}^3$
  - knot vector $T = (t_0 \leq \ldots \leq t_{m+3})$
- Find: cubic spline curve $\mathbf{s}(u)$ over T with $\mathbf{s}(t_{i+3}) = \mathbf{c}_i$ (interpolating at the knots)

Possible additional conditions for the sake of uniqueness:

- Specification of the tangent at the endpoints
- In the case of closed curves: same 1st and 2nd derivatives at the endpoints t_3 and t_m
- Natural end conditions: vanishing 2nd derivatives at the endpoints

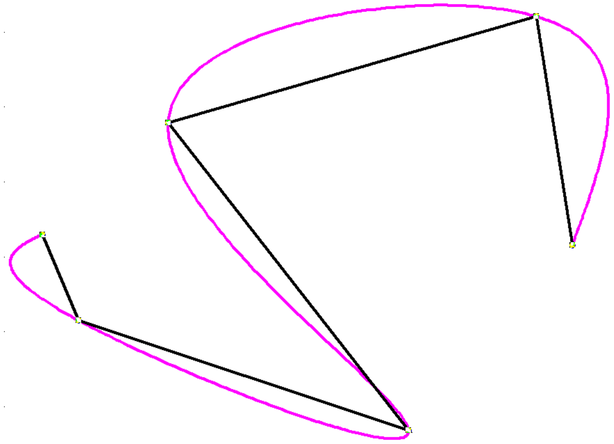
Special Case: Natural Cubic Splines

- Endpoint interpolation
- Uniform knot vector
  $T = (0, 0, 0, 0, 1, \ldots, m-3, m-2, m-2, m-2, m-2$

$$\begin{pmatrix} 3/2 & -1/2 & & & \\ 1/4 & 7/12 & 1/6 & & \\ & 1/6 & 4/6 & 1/6 & \\ & & & \cdots & \end{pmatrix} \begin{pmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_{m-2} \end{pmatrix} = \begin{pmatrix} \mathbf{c}_0 \\ \mathbf{c}_1 \\ \vdots \\ \mathbf{c}_{m-3} \end{pmatrix}$$

and $\mathbf{d}_0 = \mathbf{c}_0$, $\mathbf{d}_{m-1} = \mathbf{c}_{m-3}$.

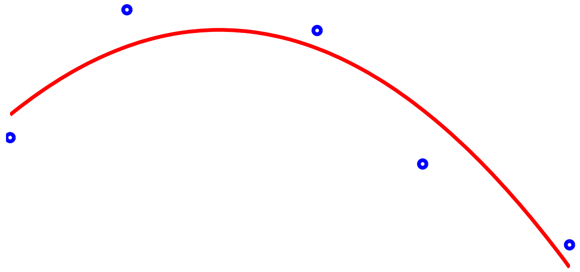
Natural cubic B–spline over uniform knot vector



B–Spline Approximation

Solve the overdetermined system of linear equations with the method of the Gaussian normal equations (for minimizing the error $||A\mathbf{d} - \mathbf{c}||^2$):

$$||A\mathbf{d} - \mathbf{c}||^2 \to \min \quad \Leftrightarrow \quad \mathbf{d} = (A^T A)^{-1} A^T \mathbf{c}$$



LSS-approximating cubic B–spline curve; uniform knot sequence

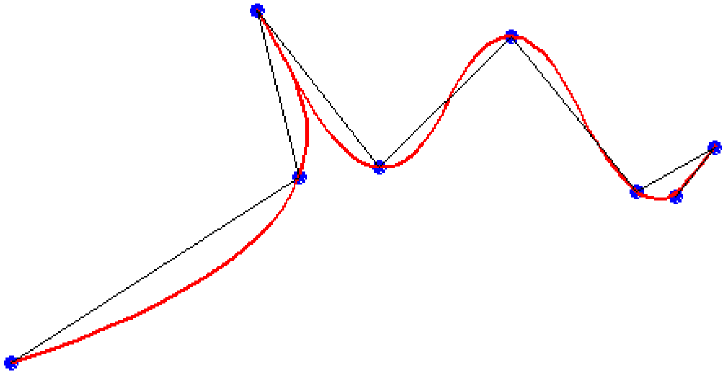
Parametrization I

How to choose parameter values x_i for data points $\mathbf{c}_i$?

- Uniform (equidistant) parametrization ($x_i = i$) - disregards the position of $\mathbf{c}_i$
- Take the positions of the data points into account, choose the distance of the parameters proportional to the distance of $\mathbf{c}_i$:

$$s_i = ||\mathbf{c}_i - \mathbf{c}_{i-1}||^p, \quad i = 1, ..., m-1, \quad p \in [0, 1]$$

$$x_0 = 0; \quad x_{i+1} = x_i + s_{i+1}, \quad i = 1, ..., m-1$$



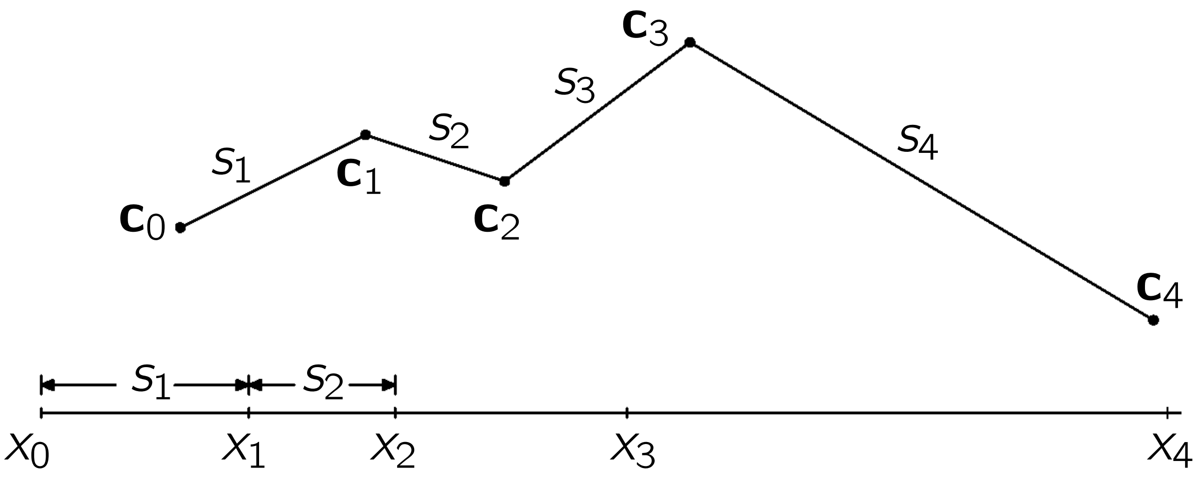
Wiggling interpolation curve (*e.g.* loops) due to parametrization ($x_i = i$)

Parametrization II

$s_i = ||\mathbf{c}_i - \mathbf{c}_{i-1}||^p, \quad i = 1, ..., m-1, \quad p \in [0,1]$

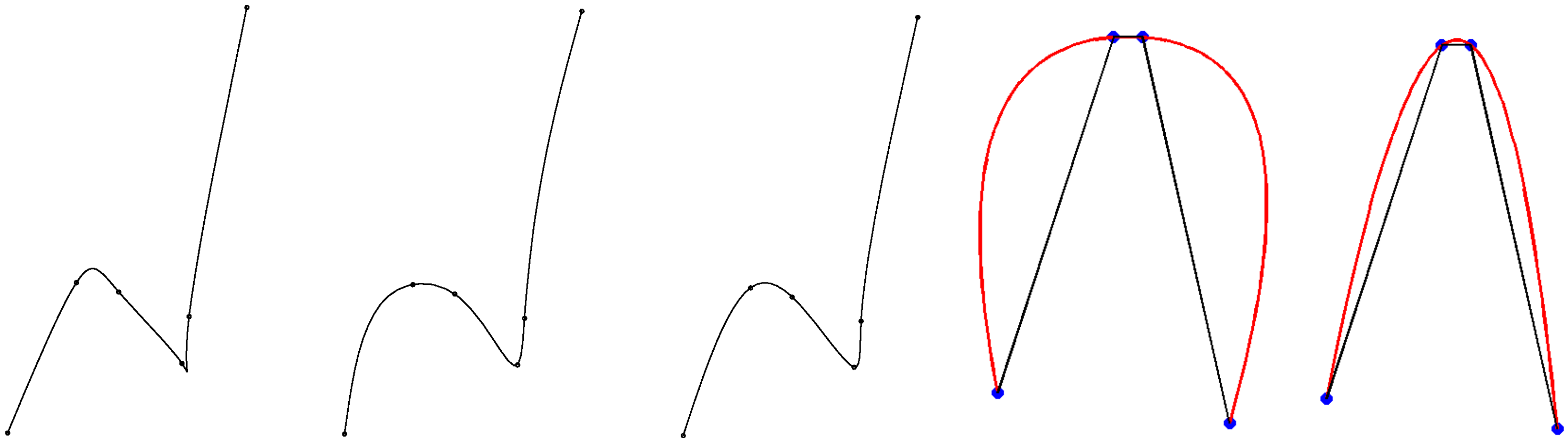
The parametrization is called

- uniform for $p = 0$
- centripetal for $p = 1/2$ – simulates a driving car through the points
- chordal for $p = 1$



Chordal parametrization

Influence of Parametrization



uniform chordal centripetal chordal centripetal

B–spline interpolation with different parametrization

Global Interpolation / Approximation

Interpolation with B–splines is global: Changing the position of s single data point changes the shape of the interpolating curve globally