Image Segmentation with Active Contours

Subproject 2: Coupling evolving level sets with curves and surfaces

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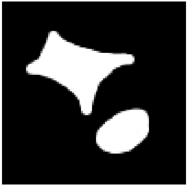
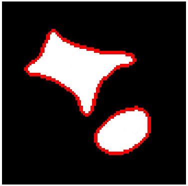
Introduction



initial image



detected object boundary

The Snake model

Let $\Omega := [-1, 1] \times [-1, 1]$. Let

$$u : \Omega \to \mathbf{R} .$$

be some given image. The boundary of the object in the image is assumed to be parametrized by a curve

$$C : [0, 1] \to \Omega .$$

The Snake model

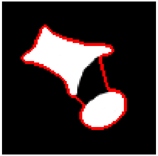
The *Snake model* (Kass et al., 1988) assumes C to be a minimizer of the energy functional

$$I_{\alpha,\beta,\gamma}^{\text{Snake}}(C) := \underbrace{\alpha \int_0^1 |C'(q)|^2 \, dq}_{\text{1st order smoothness}} + \underbrace{\beta \int_0^1 |C''(q)|^2 \, dq}_{\text{2nd order smoothness}} - \underbrace{\gamma \int_0^1 \left|\nabla I\big(C(q)\big)\right| dq}_{\text{curve position}} .$$

The Snake model

Issues:

- Many parameters (α, β, γ).
- The model has to be adapted to detect boundaries of multiple objects (topology handling).



Adaptation of the Snake model

Caselles et. al. (1997) propose a simplification of the Snake model:

$$I_{\alpha,\beta,\gamma}^{\text{simple Snake}}(C) := \underbrace{\alpha \int_0^1 |C'(q)|^2 \, dq}_{\text{1st order smoothness}} + \underbrace{\gamma \int_0^1 g\left(\left|\nabla I\left(C(q)\right)\right|\right)^2 dq}_{\text{curve position}},$$

where $g : [0, \infty[\to]0, \infty[$ strictly decreasing and $g(r) \to 0$ as $r \to \infty$.

Adaptation of the Snake model

Motivation:

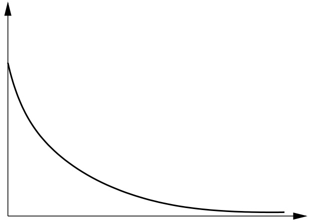
- They derive an Active Contour model from the simplified functional.
- Less parameters (α, γ, g?).
- Regularity is still preserved.

The edge detector g

The function

$$g : [0, \infty[\to]0, \infty[\,,$$

strictly decreasing and $g(r) \to 0$ as $r \to \infty$, is called *edge detector*.



The edge detector g

It is evaluated at the absolut value of the image gradient, i.e. $g(|\nabla I|)$. This yields the following properties:

$$\text{large image gradients } (\cong \text{ edges}) \longleftrightarrow g(|\nabla I|) \approx 0$$
$$\text{small image gradients} \longleftrightarrow g(|\nabla I|) > 0$$

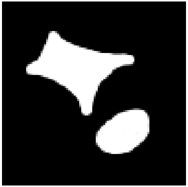


image data I

edge detector $g(|\nabla I|)$

The edge detector g

Examples:

$$g(|\nabla I|) = \frac{1}{1 + |\nabla I|^p}, \quad p \geq 1,$$

$$g(|\nabla I|) = e^{\eta |\nabla I|^2}, \quad \eta > 0.$$

For simpler notation define

$$g(I) : \Omega \to \mathbf{R}, \quad g(I)(x) := g\left(\left|\nabla I_\sigma(x)\right|\right),$$

where I_σ is a smoothed version of I (e.g. obtained by Gaussian filtering of I with parameter σ). Then $g\left(\left|\nabla I_\sigma(x)\right|\right) = g(I)(x)$.

The Active Contour model

Caselles et. al. (1997) show that minimization of $I_{\alpha,\beta,\gamma}^{\text{simple Snake}}$ is essentially equivalent to computing the steady state of the evolution

$$\frac{\partial C(\tau)}{\partial \tau} = g(I)\kappa(\tau)\mathbf{n}(\tau) - \big(\nabla g(I) \cdot \mathbf{n}(\tau)\big)\mathbf{n}(\tau)\,,$$
$$C(0) = C^0\,,$$

where κ is the curvature of C, $\mathbf{n}$ the outer unit normal and C^0 some initial contour enclosing the to be detected boundary.

Example

$\tau = 20$

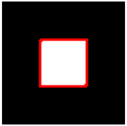
$\tau = 40$

$\tau = 60$

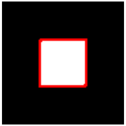
$\tau = 80$

$\tau = 100$

$\tau = 120$



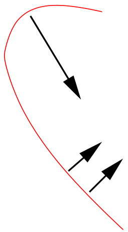
$\tau = 140$



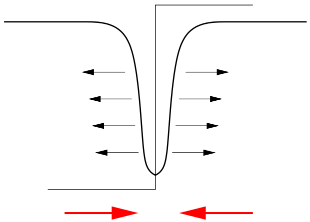
$\tau = 160$

Interpretation of the Active Contour model

$$\frac{\partial C(\tau)}{\partial \tau} = \underbrace{g(I)\kappa(\tau)\mathbf{n}(\tau)}_{\text{curvature term}} \underbrace{-(\nabla g(I)\cdot \mathbf{n}(\tau))\mathbf{n}(\tau)}_{\text{attraction term}}$$



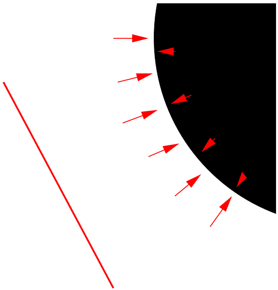
curvature term



attraction term

Degeneration of the evolution speed

- If the curve C is curvature $\kappa = 0$ the curvature term degenerates. The curvature term points outward, if the evolving curve becomes concave.
- The attraction term gets very small if the distance to the boundary of the object is large.



Balloon force

A very small evolution force leads to practical problems (slow evolution, detection of non-convex objects). Thus Caselles et. al. (1993, 1997) propose the use of a so called *balloon force* in the Active Contour model:

$$\begin{aligned} \frac{\partial C}{\partial \tau} &= g(I)(c+\kappa)\mathbf{n} - \big(\nabla g(I) \cdot \mathbf{n}\big)\mathbf{n}\,, \\ C(0) &= C^0\,, \end{aligned} \tag{E}$$

(explicit formulation), where $c < 0$ (balloon parameter).

Introduction

Active Contours

Level set evolution

FE implementation

Examples

Level set formulation

Let $u : [0, \infty[\times \Omega \to \mathbf{R}$ and assume that

$$\{C(\tau, t) : 0 \leq t \leq 1\} = \{x \in \Omega : u(\tau, x) = 0\},$$

i.e. the curve C parametrizes the zero level set of u. Then we have the equalities

$$\frac{\partial C}{\partial \tau} = -\frac{\partial u}{\partial \tau} \frac{1}{|\nabla u|} \frac{\nabla u}{|\nabla u|},$$

$$\kappa = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right),$$

$$\mathbf{n} = -\frac{\nabla u}{|\nabla u|}.$$

Level set formulation

We rewrite (E) to

$$\begin{aligned} \frac{\partial u}{\partial \tau} &= |\nabla u| g(I)\Big(c + \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big)\Big) + \nabla g(I) \cdot \nabla u\,, \\ u(0,x) &= u^0(x) \quad \text{for all } x \in \Omega \end{aligned} \tag{I}$$

(implicit formulation), where u^0 is the initial level set function. This equation is of the mean curvature motion (MCM) type.

Reformulation for viscosity solutions

There exist results concerning existence and uniqueness of solutions of (I) in the sense of *viscosity solutions*.

Assume that $g(I)$ is defined on $\mathbf{R}^2$ (e.g. by periodic extension). Define $F : \mathbf{R}^2 \times \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^{2\times 2} \to \mathbf{R}$ by

$$
\begin{aligned}
F(x, \xi, p, A) = |p| g(I)(x) c + \\
& g(I)(x)\left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) A_{ij} + \\
& \nabla g(I)(x) p
\end{aligned}
$$

(here we implicitly sum over $i, j = 1, 2$).

Reformulation for viscosity solutions

Then (I) can be reformulated on whole $\mathbf{R}^2$ by

$$\frac{\partial u(\tau,x)}{\partial \tau} = F\left(x, u(\tau,x), \nabla u(\tau,x), D^2 u(\tau,x)\right), \qquad \text{(V)}$$
$$u(0,x) = u^0(x).$$

for $\tau > 0$ and $x \in \mathbf{R}^2$.

Definition of viscosity solutions

Definition (viscosity subsolution)

Assume $u \in C([0, T[\times \mathbf{R}^2)$, $T > 0$. Then u is called a *viscositiy subsolution* of (V), if for any function $\varphi \in C^2(\mathbf{R} \times \mathbf{R}^2)$ such that $u - \varphi$ has a local maximum at (τ_0, x_0)

$$\frac{\partial \varphi(\tau_0, x_0)}{\partial \tau} \leq F\left(x, \varphi(\tau, x), \nabla \varphi(\tau, x), D^2 \varphi(\tau, x)\right)$$

if $\nabla \varphi(\tau_0, x_0) \neq 0$ and

$$\frac{\partial \varphi(\tau_0, x_0)}{\partial \tau} \leq g(I)(x_0) \limsup_{p \to 0} \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) \frac{\partial^2 u(\tau_0, x_0)}{\partial\, dx_i\, dx_j^2}$$

if $\nabla \varphi(\tau_0, x_0) = 0$.

Definition of viscosity solutions

Definition (viscosity supersolution)

Assume $u \in C([0, T[\times \mathbf{R}^2)$, $T > 0$. Then u is called a *viscositiy supersolution* of (V), if for any function $\varphi \in C^2(\mathbf{R} \times \mathbf{R}^2)$ such that $u - \varphi$ has a local minimum at (τ_0, x_0)

$$\frac{\partial \varphi(\tau_0, x_0)}{\partial \tau} \geq F\left(x, \varphi(\tau, x), \nabla\varphi(\tau, x), D^2\varphi(\tau, x)\right)$$

if $\nabla\varphi(\tau_0, x_0) \neq 0$ and

$$\frac{\partial \varphi(\tau_0, x_0)}{\partial \tau} \geq g(I)(x_0) \limsup_{p \to 0} \left(\delta_{ij} - \frac{p_i p_j}{|p|^2}\right) \frac{\partial^2 u(\tau_0, x_0)}{\partial\, dx_i\, dx_j^2}$$

if $\nabla\varphi(\tau_0, x_0) = 0$.

Definition of viscosity solutions

Definition (viscosity solution)

Assume $u \in C([0, T[\times \mathbf{R}^2)$, $T > 0$. Then u is called a *viscositiy solution* of (V), if it is both a viscosity subsolution and a viscosity supersolution of (V).

Solutions of the level set equation

Theorem (Caselles et al., 1997)

Let $g(I) : \mathbf{R}^2 \to [0, \infty[$ *be such that*

$$\sup_{x \in \mathrm{R}^2} |\nabla g^{1/2}(x)| < \infty \quad \textit{and} \quad \sup_{x \in \mathrm{R}^2} |D^2 g(x)| < \infty\,.$$

Assume further that $u^0 \in BUC(\mathbf{R}^2) \cap W^{1,\infty}(\mathbf{R}^2)$.

Solutions of the level set equation

Theorem (Caselles et al., 1997)

1. *Then there exists a unique viscosity solution u of* (V) *and*

$$u \in C([0,\infty[\times \mathbf{R}^2) \cap L^\infty([0,T[, W^{1,\infty}(\mathbf{R}^2))$$

for all $T < \infty$.

2. *If* v *is a solution of* (V) *corresponding to the initial data* v^0*, then*

$$\|u(\tau,\cdot) - v(\tau,\cdot)\|_{L^\infty} \leq \|u^0 - v^0\|_{L^\infty}$$

for all $\tau \geq 0$.

Properties

Theorem (Chen et al., 1991)

Let the initial level set function $u^0 \in W^{1,\infty}(\mathbf{R}^2) \cap BUC(\mathbf{R}^2)$. *Assume that* $u(\tau, x)$ *is the solution of* (V). *Define*

$$C(\tau) := \{x \in \mathbf{R}^2 : u(\tau, x) = 0\}\,,$$
$$\mathcal{D}(\tau) := \{x \in \mathbf{R}^2 : u(\tau, x) < 0\}\,.$$

Then, $C(\tau)$ *and* $\mathcal{D}(\tau)$ *are uniquely determined by the initial sets* $C(0)$ *and* $\mathcal{D}(0)$.

Properties

Theorem (Caselles et al., 1995)

Assume that $\hat{C} : [0,1] \to \mathbf{R}^2$ *is a simple* C^2 *Jordan curve and parametrizes*

$$\{x \in \mathbf{R}^2 : g(I)(x) = 0\} = Im(\hat{C}) \,.$$

Assume further that $Dg(I) = 0$ *on* $Im(\hat{C})$*.*

Let $u^0 \in W^{1,\infty}(\mathbf{R}^2) \cap BUC(\mathbf{R}^2)$ *be* C^2 *and*

$$\hat{C} \cup interior(\hat{C}) \subseteq \{x \in \mathbf{R}^2 : u^0(x) \le 0\} \,.$$

Properties

Theorem (Caselles et al., 1995)

Assume again that $u(\tau, x)$ *is the is the solution of* (V) *and*

$$C(\tau) := \{x \in \mathbf{R}^2 : u(\tau, x) = 0\} .$$

Then

$$\lim_{\tau \to \infty} C(\tau) = \hat{C} \quad \textit{with respect to the Hausdorff distance.}$$

Weak formulation

The original equation

$$\frac{\partial u}{\partial \tau} = |\nabla u| g(I) \Big(c + \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|} \Big) \Big) + \nabla g(I) \cdot \nabla u$$

is equivalent to

$$\frac{\partial u}{\partial \tau} = |\nabla u| \nabla \cdot \Big(g(I) \frac{\nabla u}{|\nabla u|} \Big) + c g(I) |\nabla u| \,.$$

Weak formulation

Weak formulation (after division by $|\nabla u|$):

$$\int_\Omega \frac{1}{|\nabla u|}\frac{\partial u}{\partial \tau} v = \int_\Omega \nabla \cdot \left(g(I) \frac{\nabla u}{|\nabla u|} \right) v + \int_\Omega c g(I) v \,.$$

Using the trace formula we get

$$\int_\Omega \frac{1}{|\nabla u|}\frac{\partial u}{\partial \tau} v = - \int_\Omega g(I) \frac{\nabla u}{|\nabla u|} \nabla v + \int_\Omega c g(I) v \,.$$

Semi-implicit weak formulation

Finally semi-implicit discretization in time

$$\frac{\partial u}{\partial \tau} \approx \frac{u_{n+1} - u_n}{\Delta \tau}, \quad u_0 = u^0,$$

leads to

$$\int_\Omega \frac{1}{|\nabla u_n|} u_{n+1} v + \Delta\tau \int_\Omega \frac{g(I)}{|\nabla u_n|} \nabla u_{n+1} \nabla v = \int_\Omega \left(u_n + \Delta\tau c g(I)\right) v .$$

Semi-implicit FE implementation

Using finite elements $(\varphi_i)_{i\in \mathrm{N}}$ and setting $u_{n+1} = \sum_{i\in \mathrm{N}} d^i_{n+1}\varphi_i$, this yields

$$\sum_{j\in \mathrm{N}} d^j_{n+1} \underbrace{\int_\Omega \frac{1}{|\nabla u_n|}\varphi_i\varphi_j}_{=A^{ij}_n} +$$

$$\sum_{j\in \mathrm{N}} d^j_{n+1} \underbrace{\Delta\tau \int_\Omega \frac{g(I)}{|\nabla u_n|}\nabla\varphi_i\nabla\varphi j}_{=B^{ij}_n} =$$

$$\underbrace{\int_\Omega \left(u_n + \Delta\tau c g(I)\right)\varphi_i}_{f^i_n} .$$

Semi-implicit FE implementation

Thus we solve the linear equation

$$(A_n + B_n)d_{n+1} = f_n$$

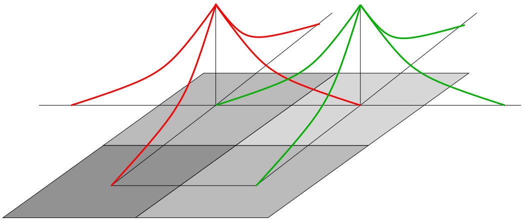
in every time step. The n-th solution u_n is given by

$$u_n = \sum_{i \in \mathrm{N}} d_n^i \varphi_i \,.$$

Remarks

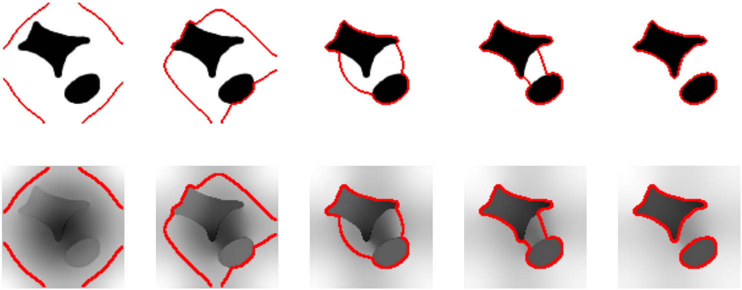
- Complexity:

$$\#\text{ degrees of freedom} \quad \sim \quad \#\text{ pixels}$$



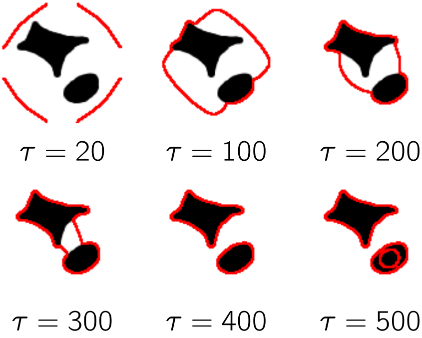
bilinear elements on 2 pixels

Example 1

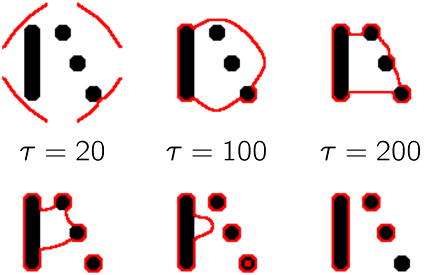


$\tau = 20$ $\tau = 100$ $\tau = 200$ $\tau = 300$ $\tau = 400$

Example 2



Example 3



$\tau = 20$ $\tau = 100$ $\tau = 200$

$\tau = 300$ $\tau = 400$ $\tau = 500$