

Hybrid Curve Fitting

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Overview

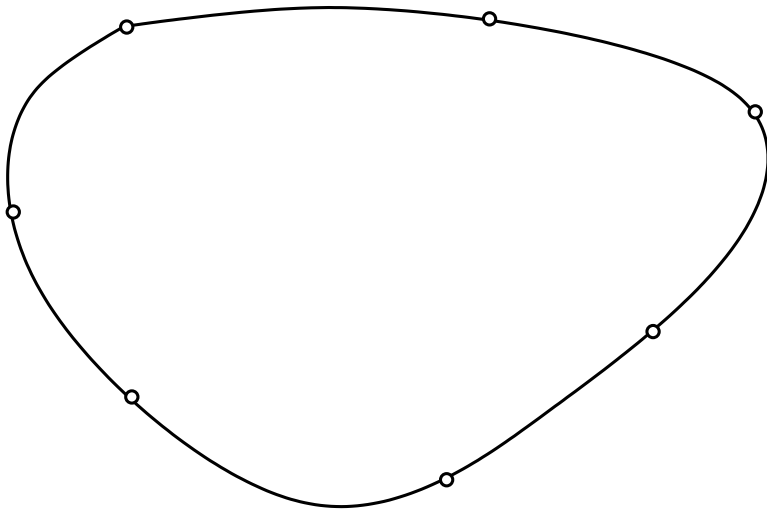
- Curve Fitting – The Problem
- Parametric Curves
- Abstract Curve Fitting as an Evolution
- Parametric Curves Revisited
- Implicit Curves
- Evolution of Hybrid Objects
- Conclusion

Curve Fitting – The Problem

Given: $\{P_j\}_{j=1..N}$



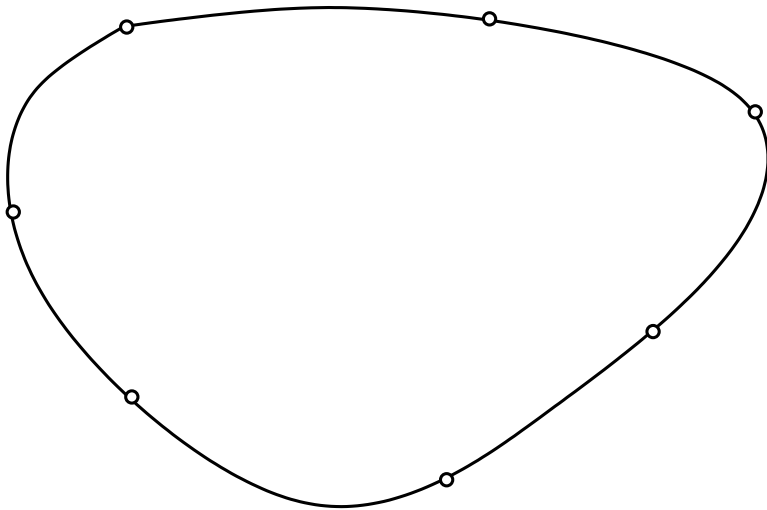
Curve Fitting – The Problem



Given: $\{\mathbf{P}_j\}_{j=1..N}$

Result: Curve \mathcal{C}

Curve Fitting – The Problem



Given: $\{\mathbf{P}_j\}_{j=1..N}$

Result: Curve \mathcal{C}

Such that:

$$\min \sum_{j=1}^N \|\mathcal{C} - \mathbf{P}_j\|^2$$

Parametric Curves

$$\mathbf{x}(u) := \sum_{i=0}^n \varphi_i(u) \mathbf{b}_i$$

$u \in [0, 1]$

φ_i Basis functions

\mathbf{b}_i coefficients ($\in \mathbb{R}^2$)

Parametric Curves

$$\mathbf{x}(u) := \sum_{i=0}^n \varphi_i(u) \mathbf{b}_i$$

$u \in [0, 1]$
 φ_i Basis functions
 \mathbf{b}_i coefficients ($\in \mathbb{R}^2$)

$$\min \sum_{j=1}^N \|\mathcal{C} - \mathbf{P}_j\|^2 = \min \sum_{j=1}^N \left\| \sum_{i=0}^n \varphi_i(u_j) \mathbf{b}_i - \mathbf{P}_j \right\|^2$$

Parametric Curves

$$\mathbf{x}(u) := \sum_{i=0}^n \varphi_i(u) \mathbf{b}_i$$

$u \in [0, 1]$
 φ_i Basis functions
 \mathbf{b}_i coefficients ($\in \mathbb{R}^2$)

$$\min \sum_{j=1}^N \|\mathcal{C} - \mathbf{P}_j\|^2 = \min_{u_j, \mathbf{b}_i} \sum_{j=1}^N \left\| \sum_{i=0}^n \varphi_i(u_j) \mathbf{b}_i - \mathbf{P}_j \right\|^2$$

Non-linear optimization problem in $N + 2(n + 1)$ unknowns!

Parametric Curves

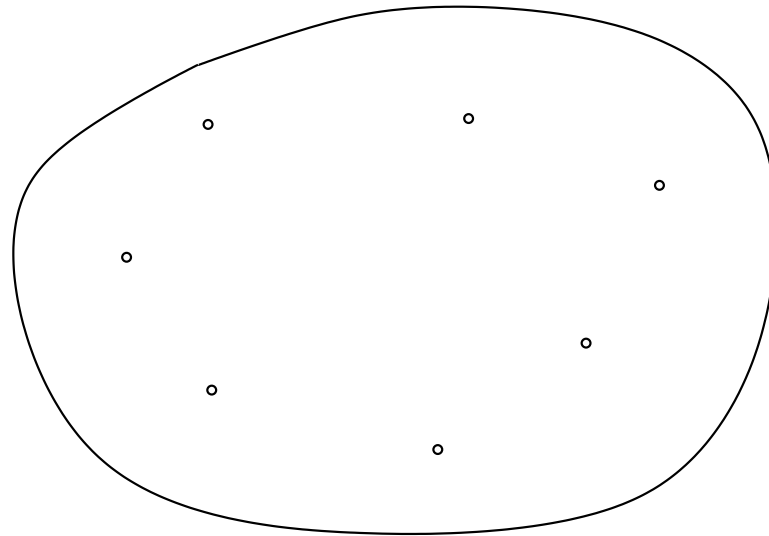
- Gauss-Newton for u_j, \mathbf{b}_i (Speer, Kuppe, Hoschek, 1998)
- Alternating method: Fix $\{u_j^*\}_{j=1..N}$ solve for $\{\mathbf{b}_i\}_{i=0..n}$ update the $\{u_j^*\}_{j=1..N}$. (Hoschek, 1993, Rogers 1989)

- TDM
$$\min \sum_{j=1}^N \left\| (\mathcal{C} - \mathbf{P}_j) \mathbf{N}_j \right\|^2$$

(Blake & Isard, 1998)

- SDM: using quadratic approximant of distance function (Pottmann, 2004)

Abstract Curve Fitting as an Evolution

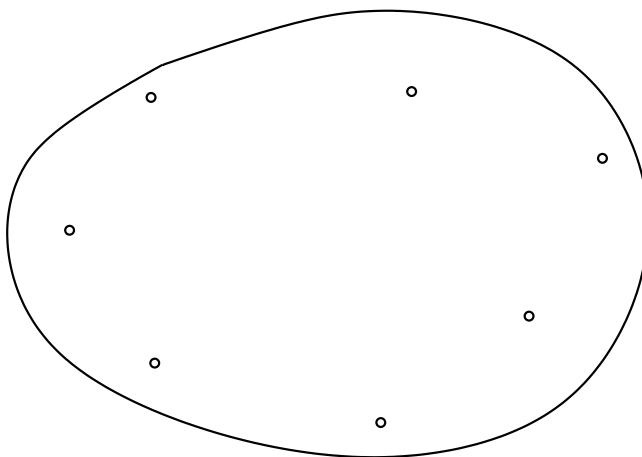


$$C_{t_0}(p_1(t_0), p_2(t_0), \dots, p_m(t_0))$$

t : time

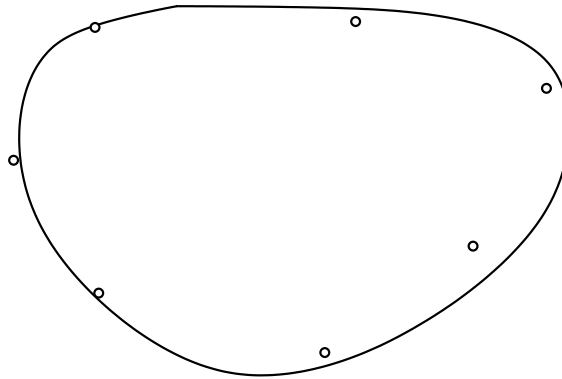
$p_1(t_0), \dots, p_m(t_0)$: shape parameters

Abstract Curve Fitting as an Evolution



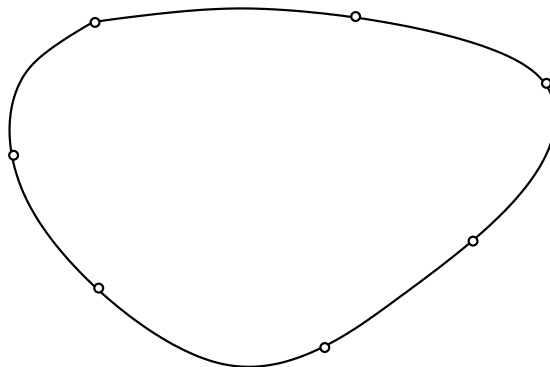
$$C_{t_1}(p_1(t_1), p_2(t_1), \dots, p_m(t_1))$$

Abstract Curve Fitting as an Evolution



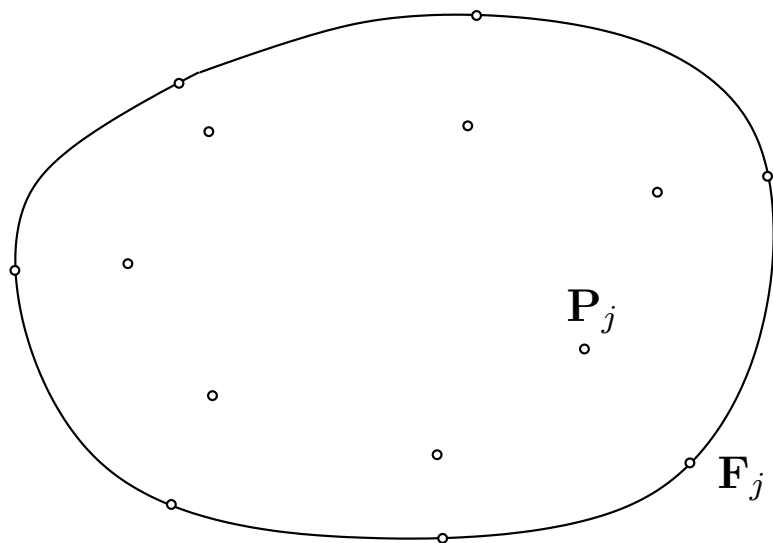
$$C_{t_2}(p_1(t_2), p_2(t_2), \dots, p_m(t_2))$$

Abstract Curve Fitting as an Evolution



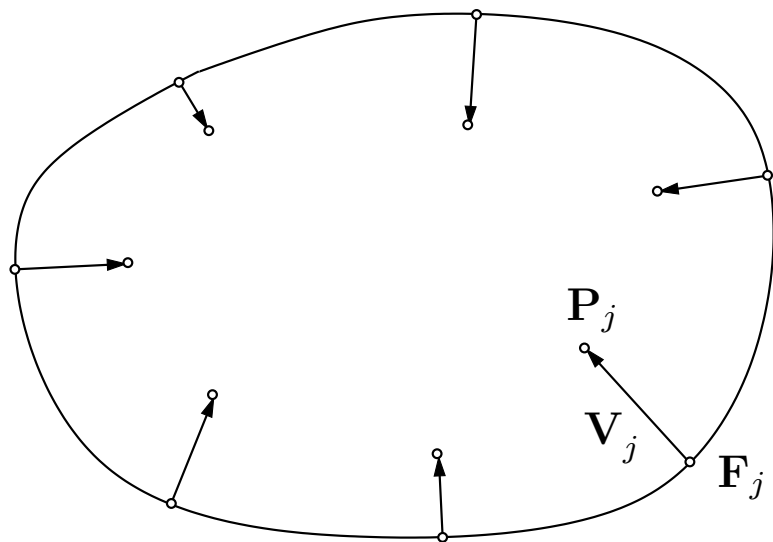
$$\mathcal{C}_{t_\infty}(p_1(t_\infty), p_2(t_\infty), \dots, p_m(t_\infty))$$

Abstract Curve Fitting as an Evolution



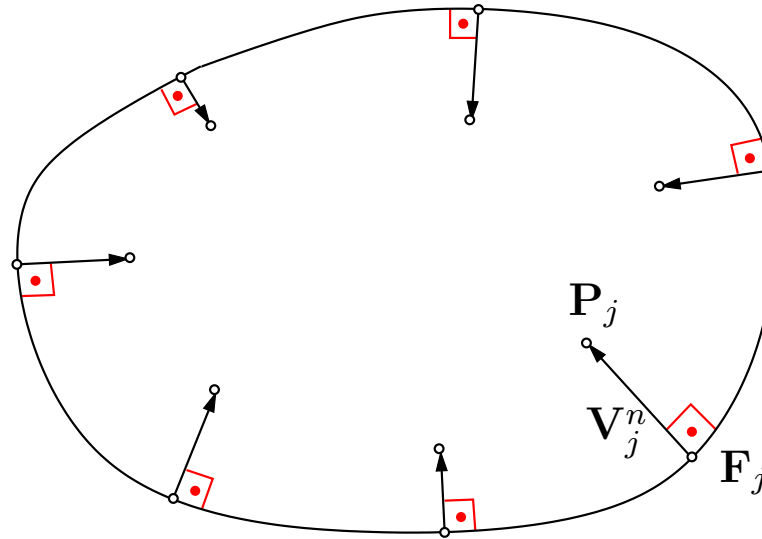
Fix a number of points F_j on the curve

Abstract Curve Fitting as an Evolution



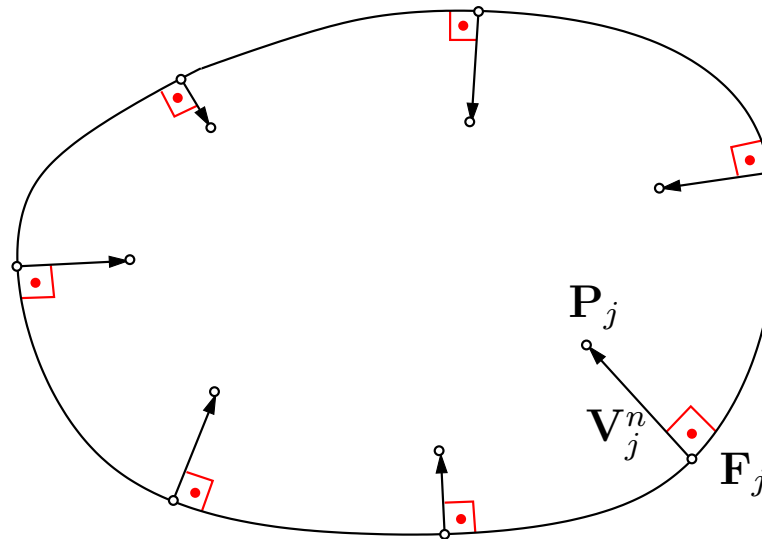
and velocities V_j in these points

Abstract Curve Fitting as an Evolution



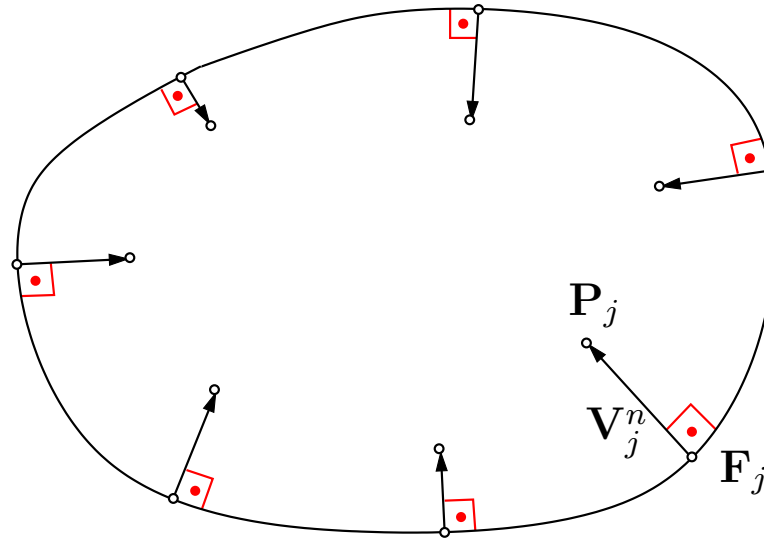
- F_j : footpoints of P_j

Abstract Curve Fitting as an Evolution



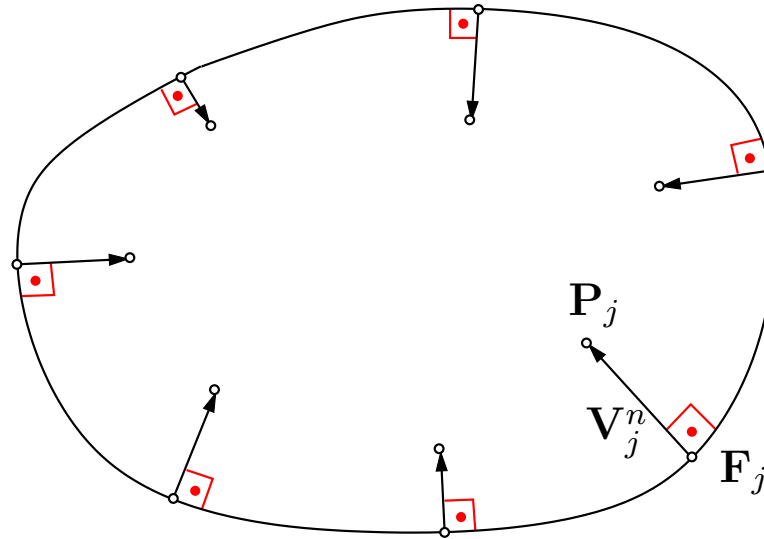
- F_j : footpoints of P_j
- V_j : same magnitude in normal direction as $P_j - F_j =: D_j$

Abstract Curve Fitting as an Evolution



$$\min \quad \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

Abstract Curve Fitting as an Evolution



$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

Parametric Curves Revisited

$$\mathbf{x}(u_j) := \sum_{i=0}^n \varphi_i(u_j) \mathbf{b}_i$$

Parametric Curves Revisited

$$\mathbf{x}_t(u_j) := \sum_{i=0}^n \varphi_i(u_j) \mathbf{b}_i(t)$$

Parametric Curves Revisited

$$\mathbf{V}_j = \dot{\mathbf{x}}_t(u_j) := \sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t)$$

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$$\mathbf{V}_j = \dot{\mathbf{x}}_t(u_j) := \sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t)$$

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

Parametric Curves Revisited

$$\mathbf{V}_j = \dot{\mathbf{x}}_t(u_j) := \sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t)$$

$$\min \sum_{j=1}^N \left\| \left(\sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t) - \mathbf{D}_j \right)^T \mathbf{N}_j \right\|^2$$

Parametric Curves Revisited

$$\mathbf{V}_j = \dot{\mathbf{x}}_t(u_j) := \sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t)$$

$$\min_{\dot{\mathbf{b}}_i(t)} \sum_{j=1}^N \left\| \left(\sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t) - \mathbf{D}_j \right)^T \mathbf{N}_j \right\|^2$$

ODE $\begin{cases} \text{linear in } \dot{\mathbf{b}}_i(t) \\ \text{nonlinear in } \mathbf{b}_i(t) \end{cases} \Rightarrow \text{solve with Euler-Cauchy}$

Implicit Curves

$$\mathcal{F} := \{\mathbf{x} \in \mathbb{R}^2 \mid f(\mathbf{x}) = 0\},$$

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$$\mathcal{F} := \{\mathbf{x} \in \mathbb{R}^2 \mid f(\mathbf{x}) = 0\},$$

with

$$f(\mathbf{x}) = \sum_{i=0}^n \psi_i(\mathbf{x})c_i$$

ψ_i Basis functions
 c_i coefficients ($\in \mathbb{R}$)

Implicit Curves

$$\mathcal{F}_t := \{\mathbf{x} \in \mathbb{R}^2 \mid f_t(\mathbf{x}) = 0\},$$

with

$$f_t(\mathbf{x}) = \sum_{i=0}^n \psi_i(\mathbf{x}) c_i(t)$$

ψ_i Basis functions

c_i coefficients ($\in \mathbb{R}$)

Implicit Curves

$$\mathbf{V}_j^n := -f_j \frac{\nabla f_j}{\|\nabla f_j\|^2}.$$

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

Implicit Curves

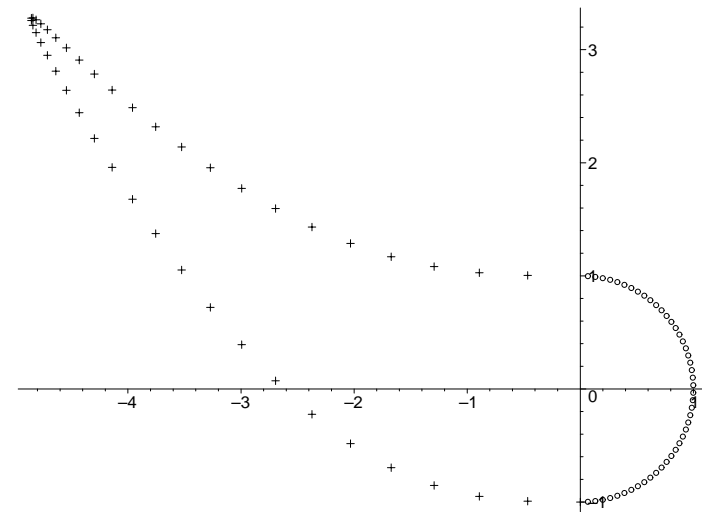
$$\mathbf{V}_j^n := -f_j \frac{\nabla f_j}{\|\nabla f_j\|^2}.$$

$$\min_{\dot{c}_i(t)} \sum_{j=1}^N \left\| \left(-\frac{f_j \nabla f_j}{\|\nabla f_j\|^2} - \mathbf{D}_j \right)^T \frac{\nabla f_j}{\|\nabla f_j\|} \right\|^2$$

ODE $\begin{cases} \text{linear in } \dot{c}_i(t) \\ \text{nonlinear in } c_i(t) \end{cases} \Rightarrow \text{solve with Euler-Cauchy}$

Hybrid Curves

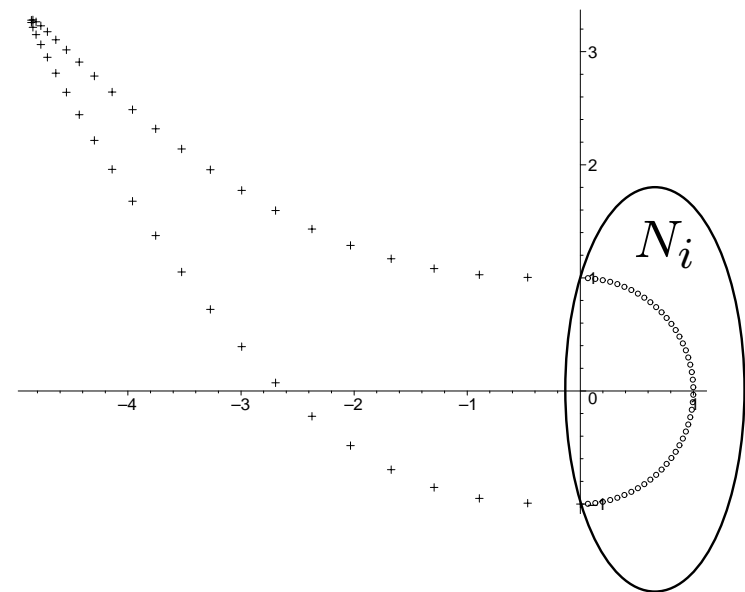
$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$



Hybrid Curves

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2 =$$

$$\min \sum_{j=1}^{N_i} \left\| -\frac{\dot{f}_j}{\|\nabla f_j\|} - \|\mathbf{D}_j\| \right\|^2$$



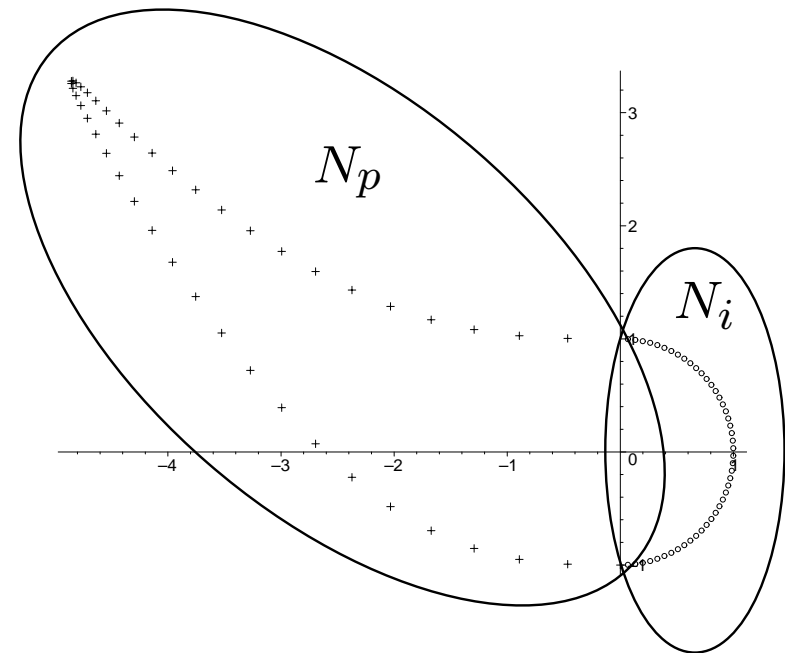
circles lie on implicit curve

Hybrid Curves

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2 =$$

$$\min \sum_{j=1}^{N_i} \left\| -\frac{\dot{f}_j}{\|\nabla f_j\|} - \|\mathbf{D}_j\| \right\|^2 +$$

$$\min \sum_{j=1}^{N_p} \left\| (\dot{\mathbf{x}}_t(u_j) - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$



circles lie on implicit curve
crosses on parametric curve

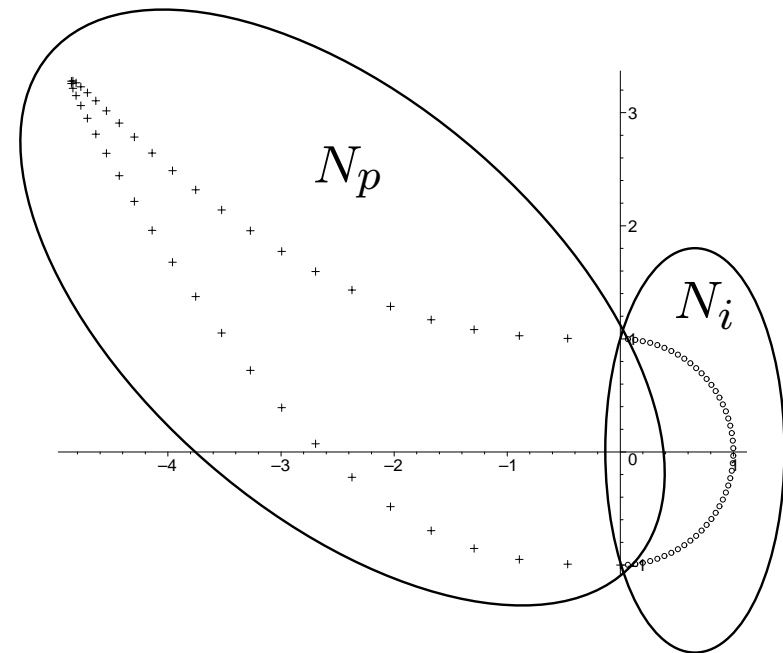
Hybrid Curves

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2 =$$

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$$\min \sum_{j=1}^{N_p} \left\| (\dot{\mathbf{x}}_t(u_j) - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

solve for \dot{c}_i, \dot{b}_i



circles lie on implicit curve
crosses on parametric curve

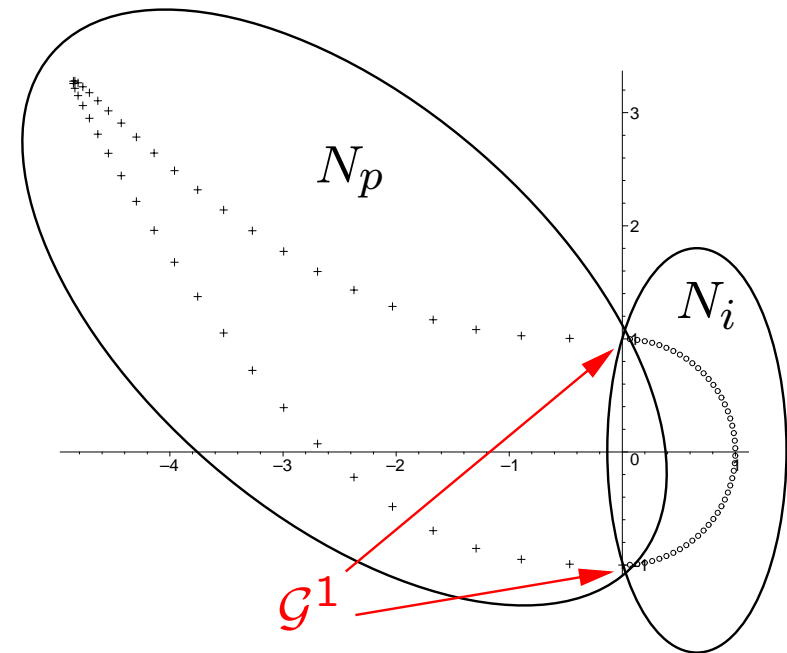
Hybrid Curves

$$\min \sum_{j=1}^N \left\| (\mathbf{V}_j - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2 =$$

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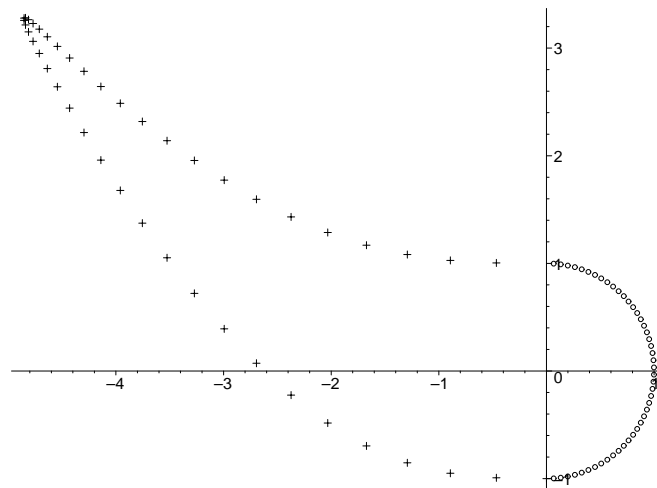
$$\min \sum_{j=1}^{N_p} \left\| (\dot{\mathbf{x}}_t(u_j) - \mathbf{D}_j)^T \mathbf{N}_j \right\|^2$$

solve for \dot{c}_i, \dot{b}_i

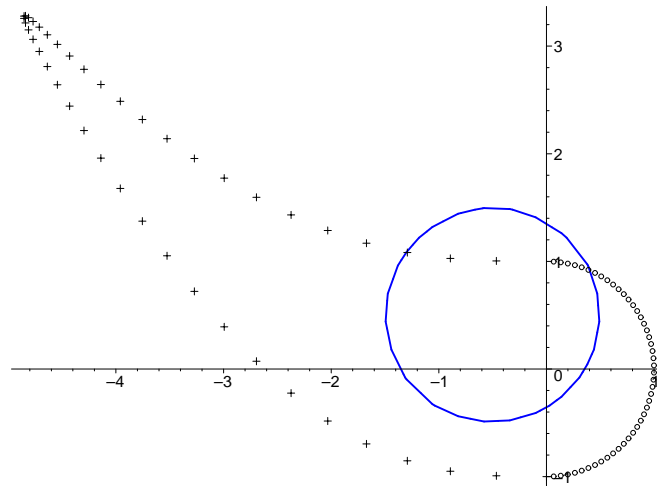


circles lie on implicit curve
crosses on parametric curve

Hybrid Curves

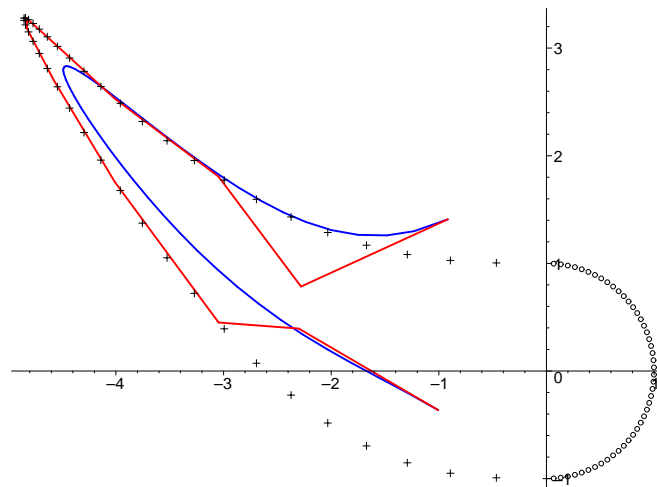


Hybrid Curves



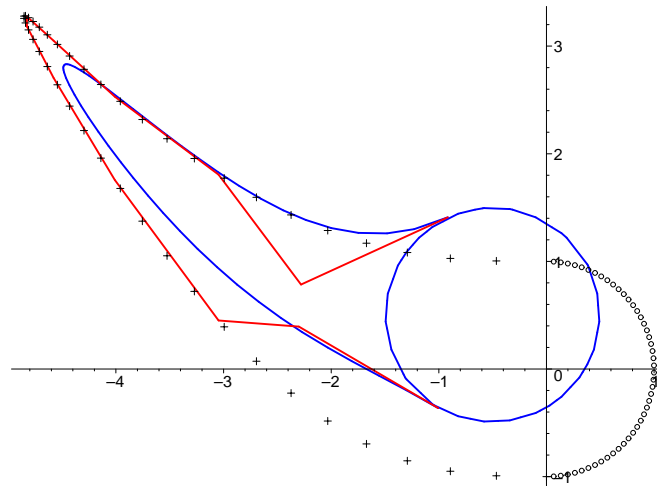
$$x_C, y_C, r$$

Hybrid Curves



$b_0, b_1, \dots, b_{n-1}, b_n$

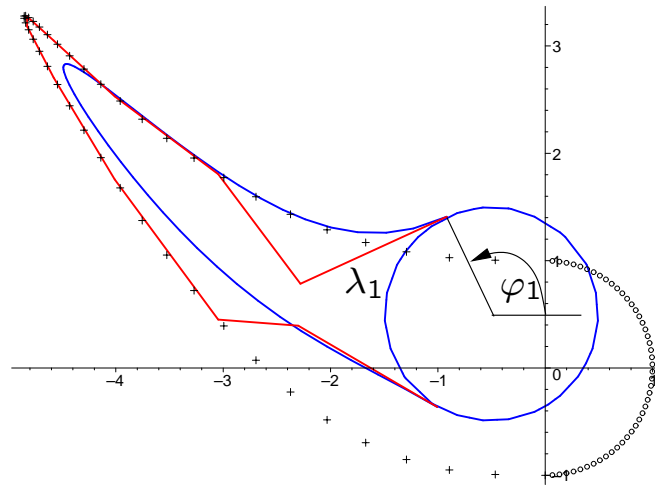
Hybrid Curves



$$\mathbf{b}_0 = (x_C + r \cos(\varphi_1), y_C + r \sin(\varphi_1))$$

$$\mathbf{b}_1 = (b_0^x - \lambda_1 \sin(\varphi_1), b_0^y + \lambda_1 \cos(\varphi_1))$$

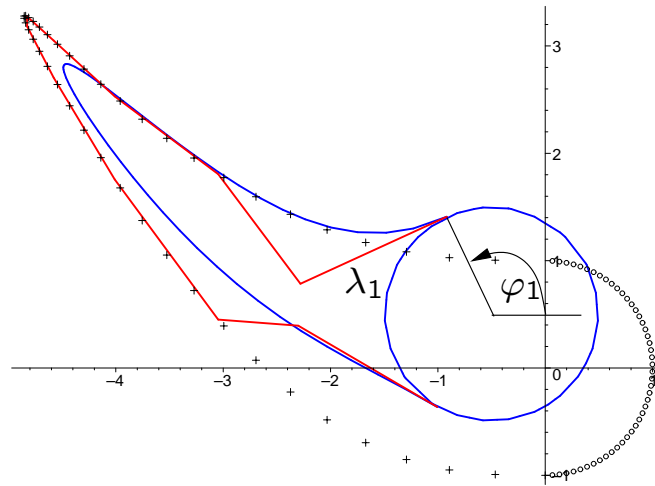
Hybrid Curves



$$\mathbf{b}_0 = (x_C + r \cos(\varphi_1), y_C + r \sin(\varphi_1))$$

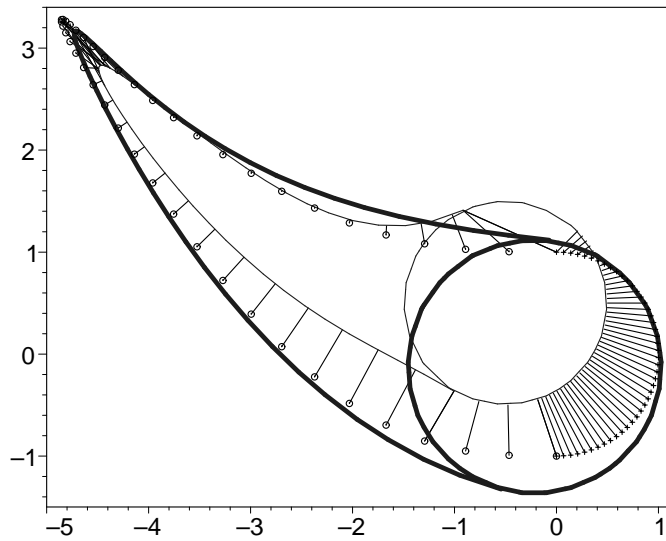
$$\mathbf{b}_1 = (b_0^x - \lambda_1 \sin(\varphi_1), b_0^y + \lambda_1 \cos(\varphi_1))$$

Hybrid Curves



$(b_2, b_3, \dots, b_{n-3}, b_{n-2}, x_C, y_C, r, \varphi_1, \varphi_2, \lambda_1, \lambda_2)$

Hybrid Curves

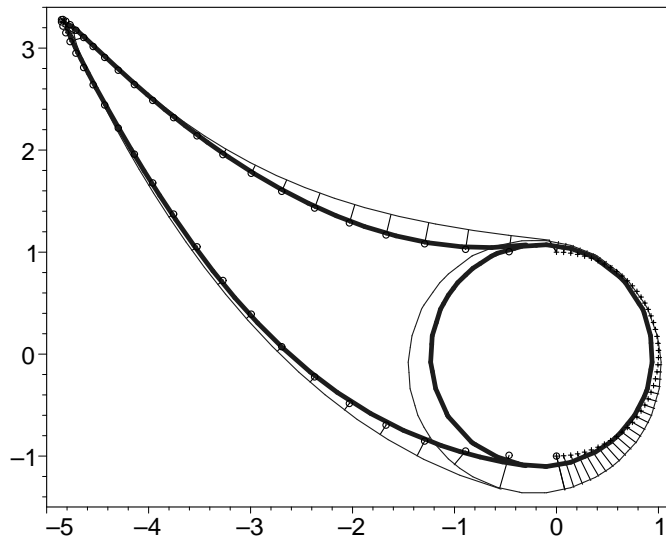


thin: old iterate

thick: new one

footpoint distances to old iterate

Hybrid Curves

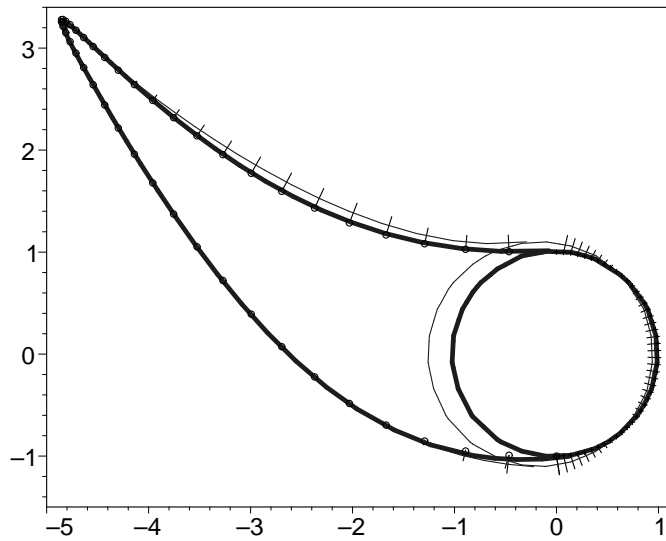


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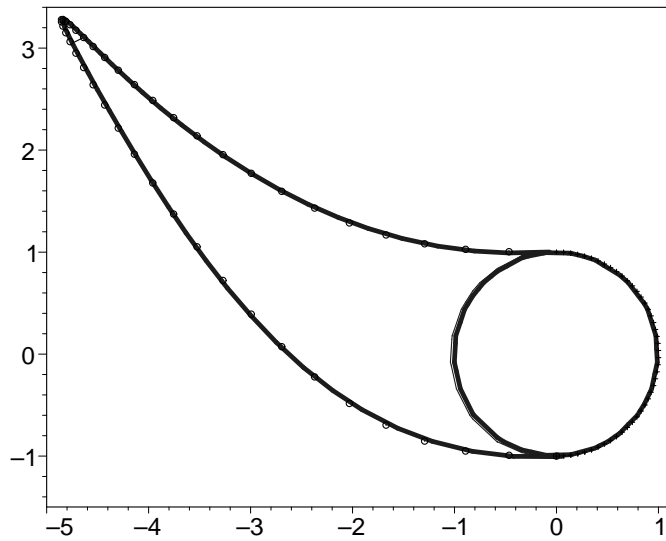


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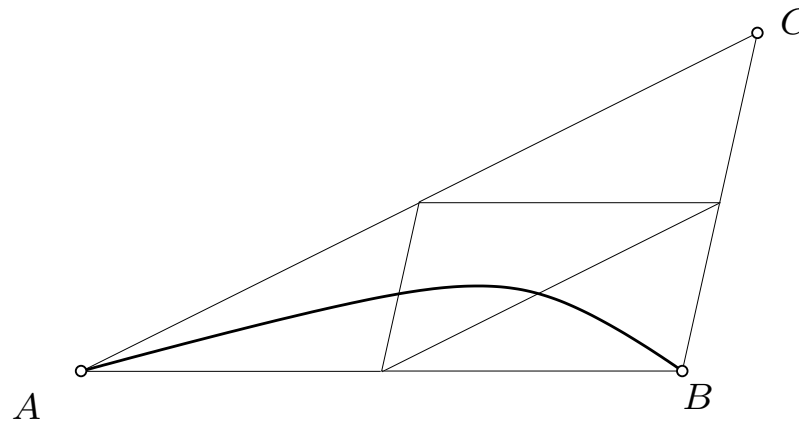


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footpoint distances to old iterate

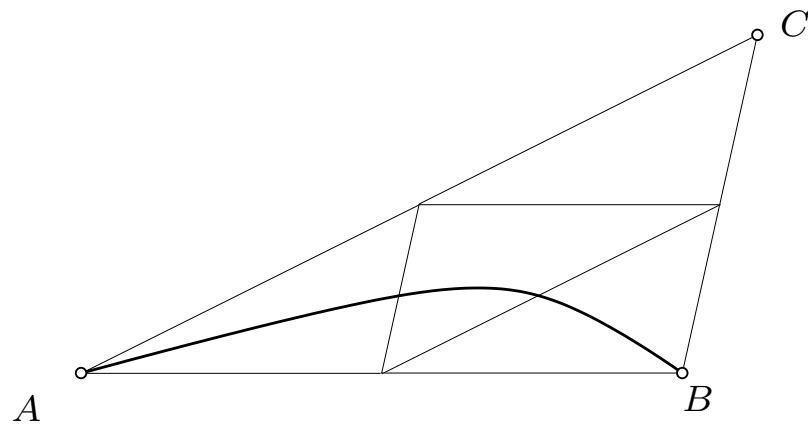
\mathcal{G}^1 continuity



$$f(\mathbf{x}) := \sum_{\substack{i,j,k \in \mathbb{N} \\ i+j+k=n}} c_{ijk} B_{ijk}^n(u, v, w)$$

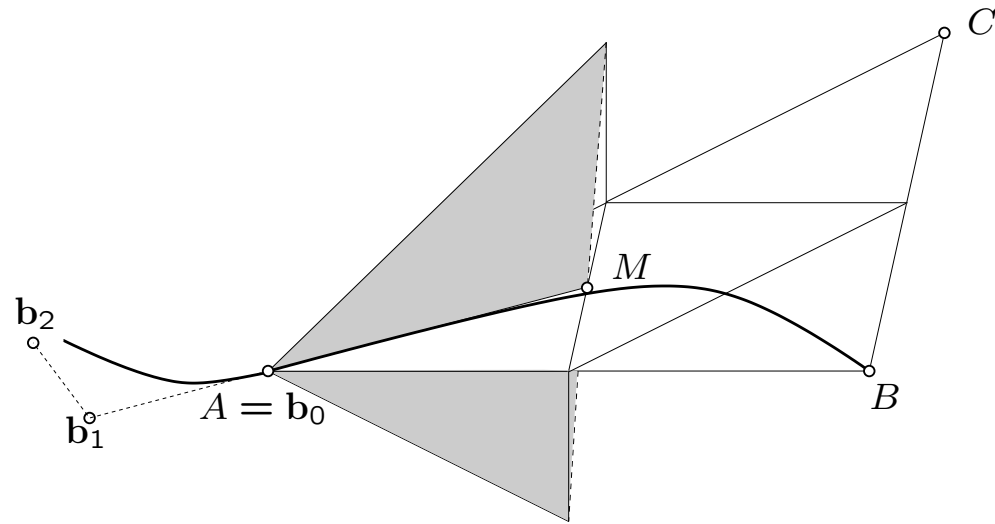
$$\text{with } B_{ijk}^n(u, v, w) := \frac{n!}{i!j!k!} u^i v^j w^k, i + j + k = n.$$

\mathcal{G}^1 continuity



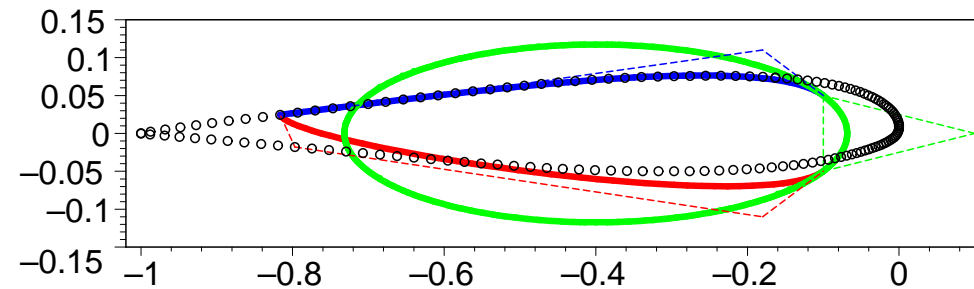
Control points in A and B set to 0.

\mathcal{G}^1 continuity



$$\mathbf{b}_1 - \mathbf{b}_0 = \lambda \left(\mathbf{b}_0 - \begin{pmatrix} M_x \\ M_y \end{pmatrix} \right)$$

Example

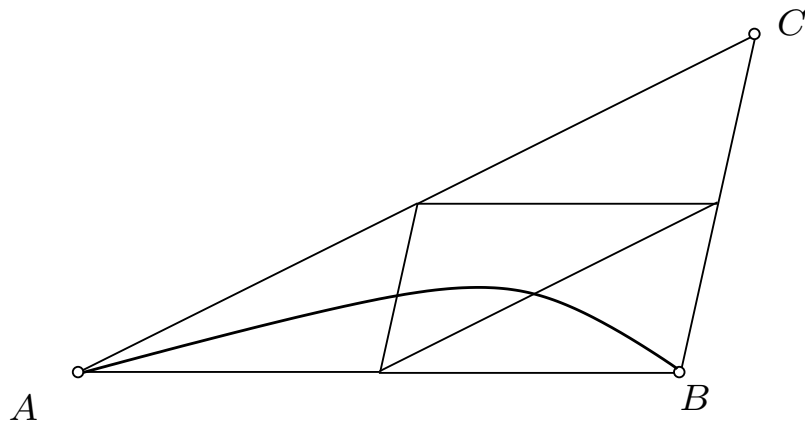


Cross-section of an airfoil and initial values
red / blue: parametric branches (\mathcal{G}^0 junction)
green: implicit curve of degree 2 (\mathcal{G}^1 junction)

Conclusion

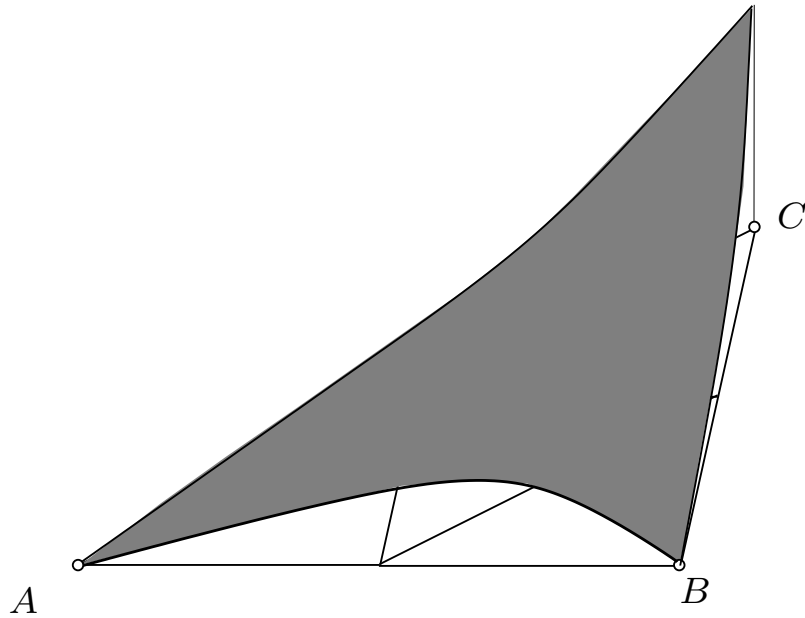
We introduced a curve fitting method that may be applied to curves in different representations. For parametric and implicit curves this was explicitly shown. Moreover, this method allows the construction of Hybrid Curves that consist of branches in different representations, as shown in some examples.

\mathcal{G}^1 continuity, revisited



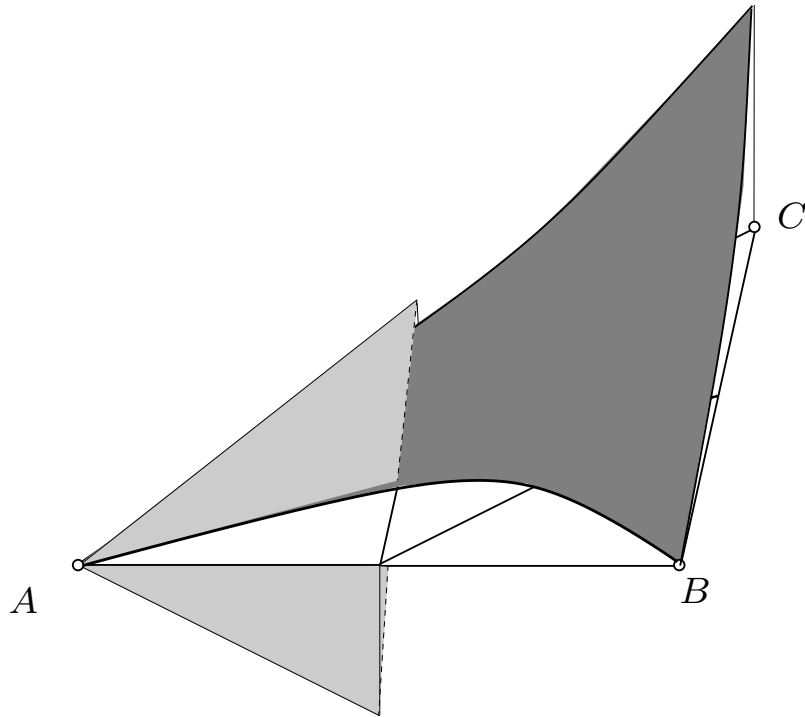
Triangular Bézier patch, the defining polynomial.

\mathcal{G}^1 continuity, revisited



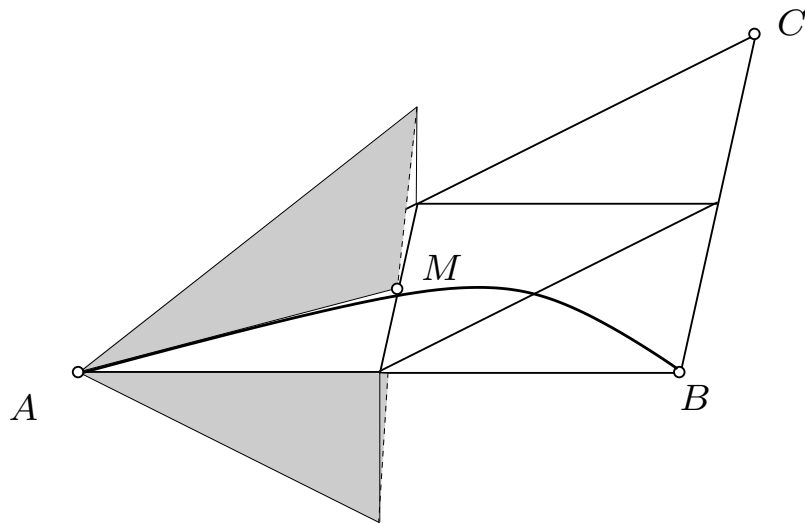
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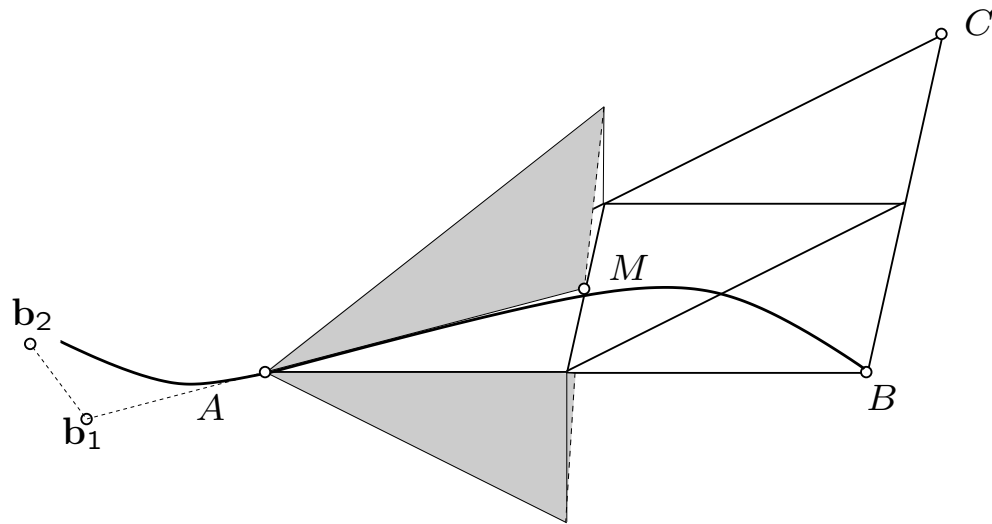
Triangular Bézier patch, the defining polynomial.

\mathcal{G}^1 continuity, revisited



Triangular Bézier patch, the defining polynomial.

\mathcal{G}^1 continuity, revisited



Triangular Bézier patch, the defining polynomial.

Euler-Cauchy

$$\min_{\dot{\mathbf{b}}_i(t)} \sum_{j=1}^N \left\| \left(\sum_{i=0}^n \varphi_i(u_j) \dot{\mathbf{b}}_i(t) - \mathbf{D}_j \right)^T \mathbf{N}_j \right\|^2$$

$$\mathbf{A}_{(\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n)} \begin{pmatrix} \dot{\mathbf{b}}_0 \\ \vdots \\ \dot{\mathbf{b}}_n \end{pmatrix} = \mathbf{r}_{(\mathbf{b}_0, \mathbf{b}_1, \dots, \mathbf{b}_n)}$$

solve for $(\dot{\mathbf{b}}_0, \dot{\mathbf{b}}_1, \dots, \dot{\mathbf{b}}_n)$,

update $(\mathbf{b}_0 + h\dot{\mathbf{b}}_0, \mathbf{b}_1 + h\dot{\mathbf{b}}_1, \dots, \mathbf{b}_n + h\dot{\mathbf{b}}_n)$.