Basic geometry for CAGD

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Overview

- Coordinates
- Geometries: Mappings
- Geometries: Invariants
- Bézier Curves

Coordinates ($d = 2$)

- Affine / Cartesian coordinates

- Barycentric coordinates

- Homogeneous coordinates

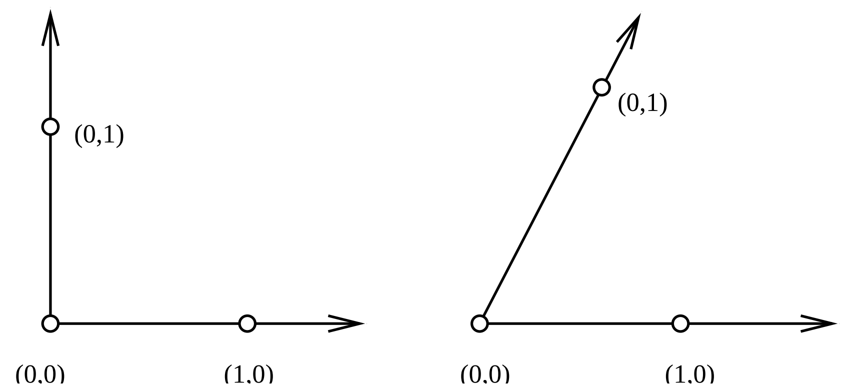
- Projective coordinates

Affine Coordinates

$$\underline{\mathbf{x}} = (\underline{x}_1, \underline{x}_2)^\top$$

Cartesian Coordinates:

- unit lengths
- orthogonal axes

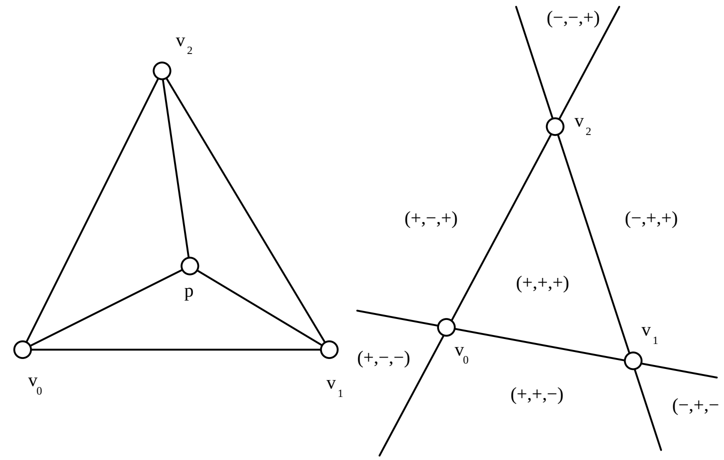


Barycentric coordinates

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: basis triangle

$$\underline{\mathbf{x}} = \xi_1 \underline{\mathbf{v}}_1 + \xi_2 \underline{\mathbf{v}}_2 + \xi_3 \underline{\mathbf{v}}_3$$

$$1 = \xi_1 + \xi_2 + \xi_3$$



$$\xi_i = \frac{\det \begin{pmatrix} 1 & 1 & 1 \\ \underline{x}_1 & \underline{v}_{i+1,1} & \underline{v}_{i+2,1} \\ \underline{x}_2 & \underline{v}_{i+1,2} & \underline{v}_{i+2,2} \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & 1 \\ \underline{v}_{i,1} & \underline{v}_{2,1} & \underline{v}_{3,1} \\ \underline{v}_{i,2} & \underline{v}_{2,2} & \underline{v}_{3,2} \end{pmatrix}} = \frac{\text{area} \triangle\, \underline{\mathbf{x}}, \underline{\mathbf{v}}_{i+1}, \underline{\mathbf{v}}_{i+2}}{\text{area} \triangle\, \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2, \underline{\mathbf{v}}_3}$$

(indices counted modulo 3)

Homogeneous coordinates

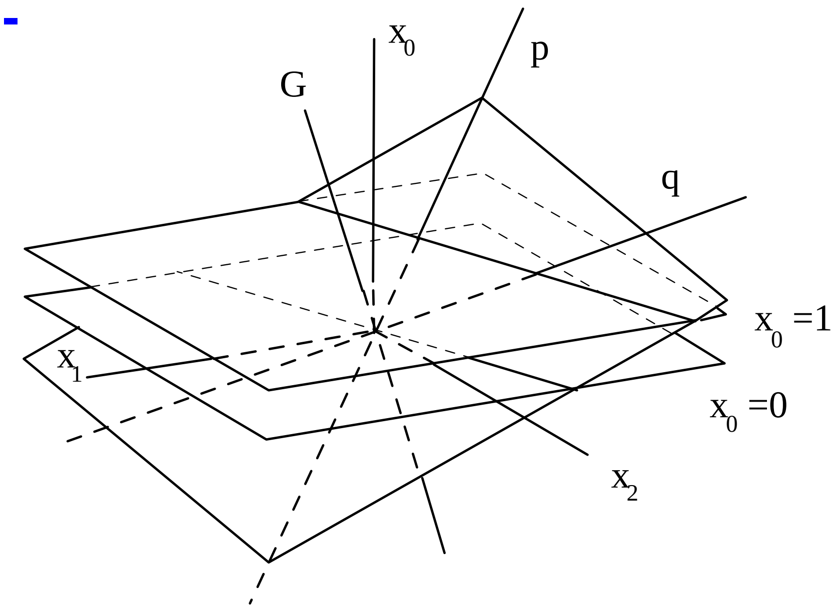
Points are identified with **1-dim sub-spaces:**

$$\tilde{\mathbf{x}} \doteq (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)^\top$$

$$1 : \underline{x}_1 : \underline{x}_2 = \tilde{x}_0 : \tilde{x}_1 : \tilde{x}_2$$

$$\underline{x}_i = \tilde{x}_i / \tilde{x}_0$$

$\tilde{x}_i = 0 \Leftrightarrow$ points at infinity



Homogeneous coordinates of hyperplanes ($d = 2$)

Lines are identified with 2-dim subspaces:

$$\tilde{\mathbf{L}} = (\tilde{L}_0, \tilde{L}_1, \tilde{L}_2)^\top$$

Equation of a line:

$$0 = \tilde{\mathbf{L}}^\top \tilde{\mathbf{x}} = \tilde{L}_0 \tilde{x}_0 + \tilde{L}_1 \tilde{x}_1 + \tilde{L}_2 \tilde{x}_2$$

Two points $\tilde{\mathbf{p}}, \tilde{\mathbf{q}}$ span the line $\tilde{\mathbf{L}} = \tilde{\mathbf{p}} \vee \tilde{\mathbf{q}}$.

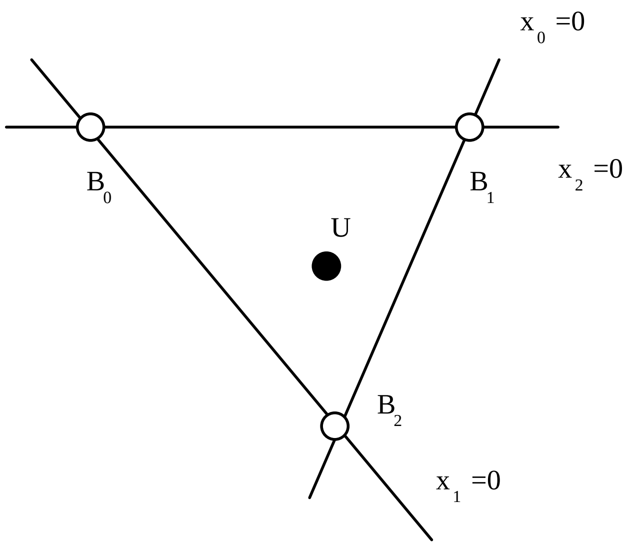
Two lines $\tilde{\mathbf{L}}, \tilde{\mathbf{K}}$ intersect in the point $\tilde{\mathbf{p}} = \tilde{\mathbf{L}} \wedge \tilde{\mathbf{K}}$

Computation of $\vee, \wedge$: usual cross-product

Projective coordinates

$$\hat{\mathbf{x}} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)^\top$$

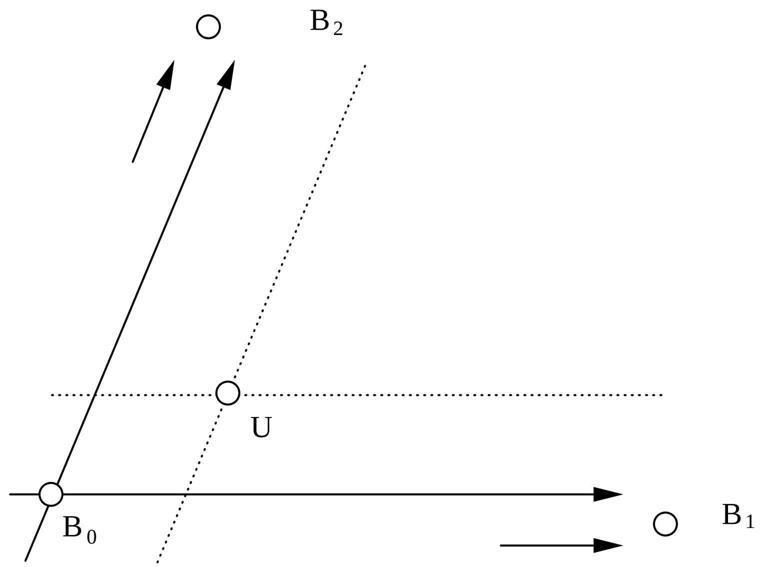
$$\hat{\mathbf{x}} \doteq A \begin{pmatrix} 1 \\ \underline{x}_1 \\ \underline{x}_2 \end{pmatrix}, \quad A \in \mathbb{R}^{3\times 3},$$ constant



$\ldots$ are determined by the $d+1$ **basis points** $\tilde{\mathrm{b}}_i$,
$\tilde{\mathrm{b}}_1 = (1, 0, 0)^\top$ etc., and the **unit point** $\mathbf{u} = (1, 1, 1)^\top$
Hyperplanes, $\vee$, $\wedge$: as for homogeneous coordinates.

Homogeneous coordinates are special projective coordinates

basis points: origin and infinite points of the axes



unit point: $\underline{\mathbf{u}} = (1, 1)^\top$

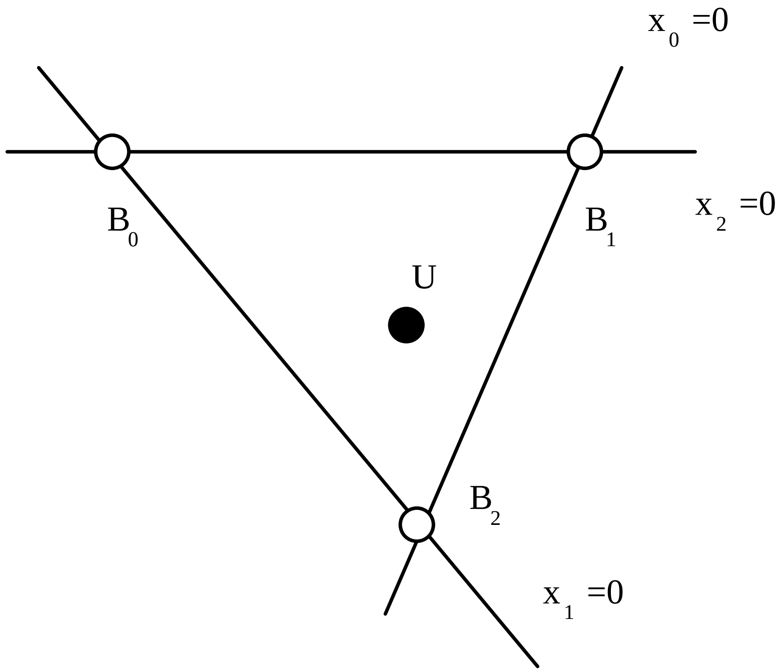
$A =$ identity matrix (or multiple thereof)

Barycentric coordinates are special projective coordinates

basis points: vertices of the domain triangle

unit point: barycenter of the domain triangle

$$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^\top \doteq (1, 1, 1)^\top$$



+ additional normalization: $\widehat{p}_0 + \widehat{p}_1 + \widehat{p}_2 = 1$
for finite points

points at infinity satisfy $\widehat{p}_0 + \widehat{p}_1 + \widehat{p}_2 = 0$

Mappings

- Projective transformations
- Affine transformations
- Euclidean similarities
- Absolute figures

Projective transformations

Points are transformed according to

$$\pi : \tilde{\mathbf{p}} \mapsto \tilde{\mathbf{p}}' = P\tilde{\mathbf{p}}$$

$\tilde{\mathbf{p}}, \tilde{\mathbf{p}}'$: homogeneous coordinates

P: non-singular matrix

Hyperplanes are transformed according to

$$\pi : \tilde{\mathbf{L}} \mapsto \tilde{\mathbf{L}}' = (P^\top)^{-1}\tilde{\mathbf{L}}$$

$\tilde{\mathbf{L}}, \tilde{\mathbf{L}}'$: homogeneous hyperplane coordinates

Affine transformations

are **special projective transformations**

$$\alpha : \tilde{\mathbf{p}} \mapsto \tilde{\mathbf{p}}' = A\tilde{\mathbf{p}} \text{ with } \begin{pmatrix} a & 0 & 0 \\ av_1 & ab_{11} & ab_{12} \\ av_2 & ab_{21} & ab_{22} \end{pmatrix}$$

[$\tilde{\mathbf{p}}, \tilde{\mathbf{p}}'$: homogeneous coordinates]

or

$$\alpha : \underline{\mathbf{p}} \mapsto \underline{\mathbf{p}}' = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \underline{\mathbf{p}}$$

[$\underline{\mathbf{p}}, \underline{\mathbf{p}}'$ affine coordinates]

$a \neq 0, b_{11}b_{22} - b_{21}b_{12} \neq 0$

translations, rotations, axial scalings

Euclidean similarities

are **special affine transformations**

$$\sigma : \tilde{\mathbf{p}} \mapsto \tilde{\mathbf{p}}' = \begin{pmatrix} a & 0 & 0 \\ av_1 & ab_{11} & ab_{12} \\ av_2 & ab_{21} & ab_{22} \end{pmatrix}$$

or

$$\sigma : \underline{\mathbf{p}} \mapsto \underline{\mathbf{p}}' = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \underline{\mathbf{p}}$$

$a \neq 0$, (b_{ij}) = constant multiple of an orthogonal matrix

uniform scaling $\circ$ translation $\circ$ rotation [$\circ$ reflection]

Absolute figures

- Line (hyperplane) at infinity $\tilde{p}_0 = 0$ = collection of all points at infinity

- "circle points at infinity"

$$\tilde{\mathbf{c}}_1 = (0, 1, i), \quad \tilde{\mathbf{c}}_2 = (0, 1, -i)$$

= common intersection of all circles:

$$(\underline{x}_1 - \underline{m}_1)^2 + (\underline{x}_2 - \underline{m}_2)^2 = r^2$$

$$(\tilde{x}_1 - \tilde{x}_0\underline{m}_1)^2 + (\tilde{x}_2 - \tilde{x}_0\underline{m}_2)^2 = \tilde{x}_0^2 r^2$$

intersection with line at infinity: $\tilde{x}_0 = 0$

$$\tilde{x}_1^2 + \tilde{x}_2^2 = 0$$

$\Rightarrow$ **Two conjugate complex points!**

Characterization of affine transformations and Euclidean similarities

affine transformations: all projective transformations that map the **line (hyperplane) at infinity** into itself

Euclidean similarities: all affine transformations that map the **circle points at infinity** into themselves

Transformation groups	**Absolute figures**
projective transformations	$\emptyset$
affine transformations	hyperplane
Euclidean similarities	hyperplane + conic (w/o real points)

Non-Euclidean geometries

correspond to other absolute figures:

Elliptic geometry: $\tilde{x}_0^2 + \tilde{x}_1^2 + \tilde{x}_2^2 = 0$ (empty conic)

Hyperbolic geometry: $\tilde{x}_0^2 + \tilde{x}_1^2 - \tilde{x}_2^2 = 0$ (real conic)

Pseudo-Euclidean geometry (Minkowsky geometry):

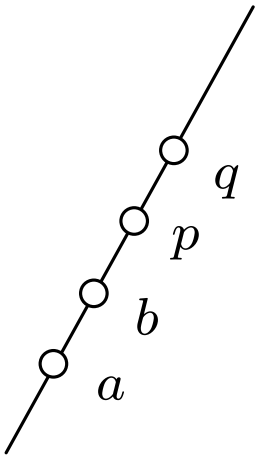
$\tilde{x}_0 = 0$ and $\tilde{c}_1 = (0, 1, 1)$ $\tilde{c}_2 = (0, 1, -1)$ (line + 2 real points)

Invariants

- Projective transformations: cross ratios
- Affine transformations: ratios
- Euclidean similarities: angles

The cross ratio

Consider 4 collinear points $\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{p}}, \tilde{\mathbf{q}}$



$$\tilde{\mathbf{a}} = \lambda_0 \tilde{\mathbf{p}} + \lambda_1 \tilde{\mathbf{q}}, \qquad \tilde{\mathbf{b}} = \mu \tilde{\mathbf{p}} + \mu_1 \tilde{\mathbf{q}}$$

The **cross ratio** $\text{cr}(a, b, p, q) = \frac{\lambda_0}{\lambda_1} : \frac{\mu_0}{\mu_1}$

is **invariant**

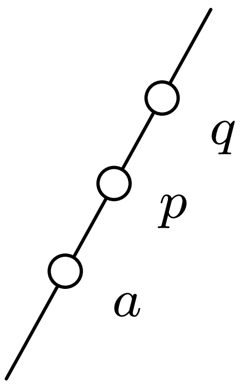
1) under **projective mappings** $\tilde{\mathbf{x}} \mapsto P\tilde{\mathbf{x}}$

2) under **renormalizations** $\tilde{\mathbf{x}} \leadsto \gamma\tilde{\mathbf{x}} \quad (\gamma \neq 0)$

The ratio

Consider 3 collinear points $\underline{\mathbf{a}}, \underline{\mathbf{p}}, \underline{\mathbf{q}}$

$$\underline{\mathbf{a}} = \lambda_0 \underline{\mathbf{p}} + \lambda_1 \underline{\mathbf{q}}, \qquad \lambda_0 + \lambda_1 = 1$$



The **ratio** ratio $(\underline{\mathbf{a}}, \underline{\mathbf{p}}, \underline{\mathbf{q}}) = \dfrac{\lambda_0}{\lambda_1}$

is invariant under affine mappings $\underline{\mathbf{x}} \mapsto \underline{\mathbf{v}} + \mathbf{A}\underline{\mathbf{x}}$

The ratio is a special cross ratio:

Consider 4 points on the line $\tilde{p}_2 = 0$, $\tilde{\mathrm{b}}$ at infinity:

$$\tilde{\mathbf{a}} = \begin{pmatrix} 1 \\ a \\ 0 \end{pmatrix} \quad \tilde{\mathbf{b}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \tilde{\mathbf{p}} = \begin{pmatrix} 1 \\ p \\ 0 \end{pmatrix} \quad \tilde{\mathbf{q}} = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix}$$

$$\tilde{\mathbf{a}} = \lambda_0 \tilde{\mathbf{p}} + \lambda_1 \tilde{\mathbf{q}} \quad \Rightarrow \quad \lambda_0 = \frac{q-a}{q-p}, \quad \lambda_1 = \frac{a-p}{q-p}$$

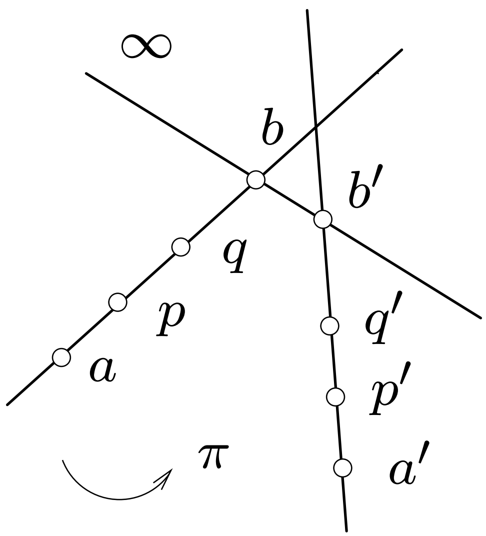
$$\tilde{\mathbf{b}} = \mu_0 \tilde{\mathbf{p}} + \mu_1 \tilde{\mathbf{q}} \quad \Rightarrow \quad \mu_0 = \frac{1}{p-q}, \quad \mu_1 = -\frac{1}{p-q}$$

$$\mathrm{cr}(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}, \tilde{\mathbf{p}}, \tilde{\mathbf{q}}) = -\frac{q-a}{a-p} = -\mathrm{ratio}(\underline{\mathbf{a}}, \underline{\mathbf{p}}, \underline{\mathbf{q}})$$

holds **whenever $\tilde{\mathrm{b}}$ is the point at infinity** of the line through $\tilde{\mathbf{a}}, \tilde{\mathbf{p}}, \tilde{\mathbf{q}}$

Absolute figure and invariants

- Projective mappings are affine mappings if and only if they leave the hyperplane at infinity invariant.
- Projective mappings, that leave the hyperplane at infinity invariant, preserve the ratio of three collinear points.



The angle between two lines

$$\tilde{\mathbf{L}}_1 = (a, \cos\varphi, \sin\varphi) \qquad \tilde{\mathbf{L}}_2 = (b, \cos\psi, \sin\psi)$$

intersect line at infinity in

$$\tilde{\mathbf{I}}_1 = (0, -\sin\varphi, \cos\varphi) \qquad \tilde{\mathbf{I}}_2 = (0, -\sin\psi, \cos\psi)$$

$$\mathrm{cr}(\tilde{\mathbf{I}}_1, \tilde{\mathbf{I}}_2, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2) = ?$$

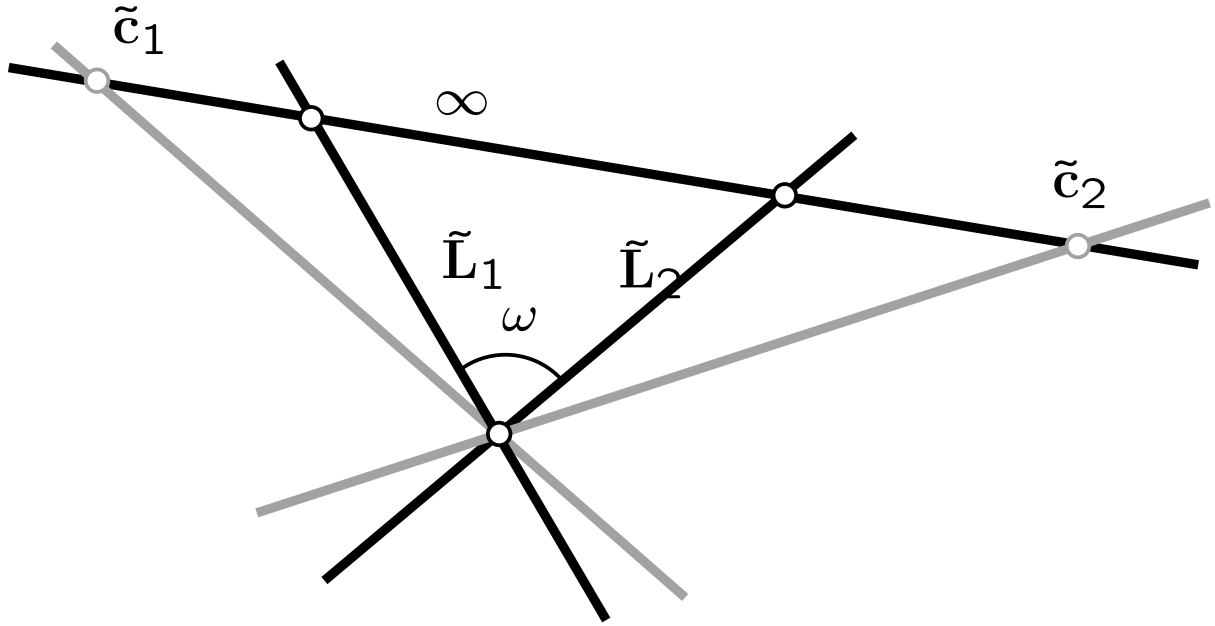
Recall: $\tilde{\mathbf{c}}_1 = (0, i, 1)$, $\quad \tilde{\mathbf{c}}_2 = (0, -i, 1)$ are the circle points at infinity

By Euler's formula:

$$2\tilde{\mathbf{l}}_1 = e^{i\varphi}\tilde{\mathbf{c}}_1 + e^{-i\varphi}\tilde{\mathbf{c}}_2, \quad 2\tilde{\mathbf{l}}_2 = e^{i\psi}\tilde{\mathbf{c}}_1 + e^{-i\psi}\tilde{\mathbf{c}}_2$$

$$\Rightarrow \mathrm{cr}(\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2) = e^{i(2\varphi - 2\psi)}$$

$$\Rightarrow \varphi - \psi = -\frac{1}{2} \ln \mathrm{cr}(\tilde{\mathbf{l}}_1, \tilde{\mathbf{l}}_2, \tilde{\mathbf{c}}_1, \tilde{\mathbf{c}}_2)$$



Invariants and absolute figures

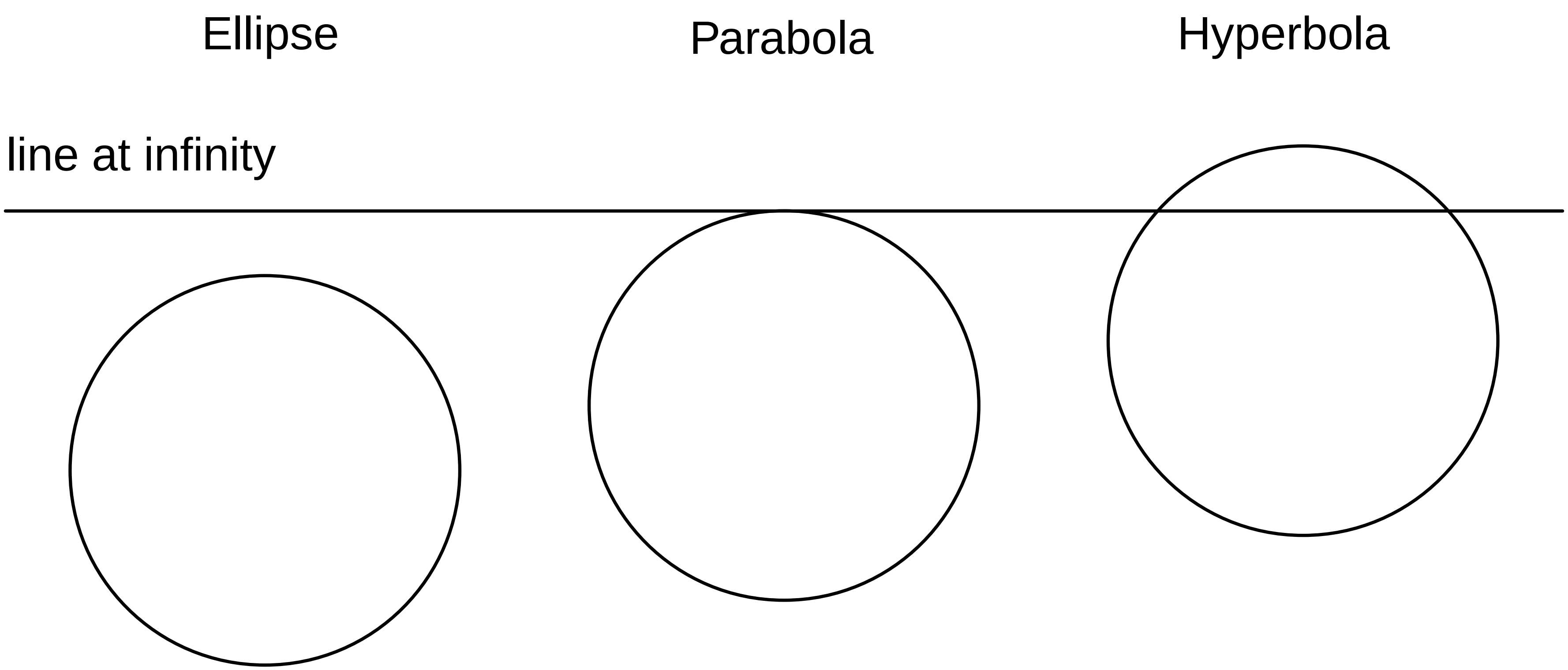
All invariants can be expressed by cross ratios and relations with respect to / additional points generated by the absolute figures.

invariants $=$ **cross ratios** $+$ **absolute figures**

Conic sections

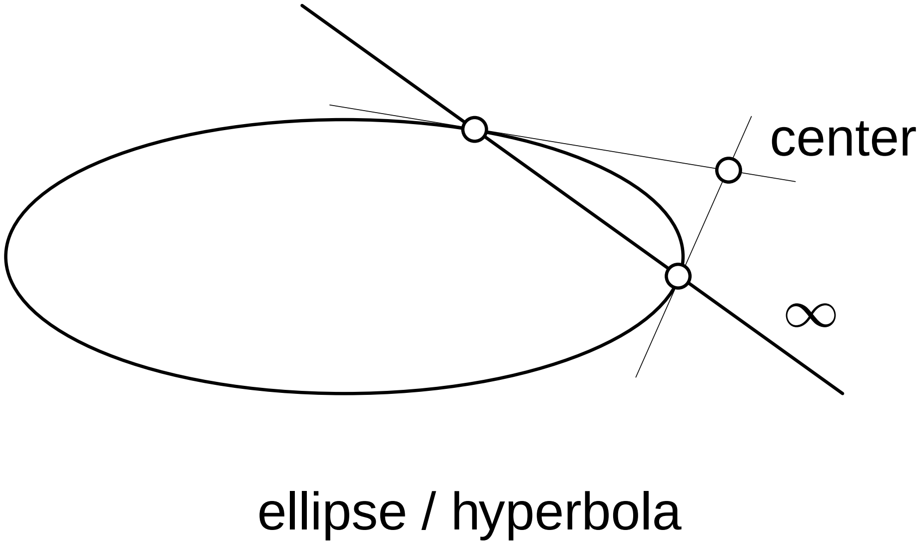
- affine classification
- affine invariants
- Euclidean invariants

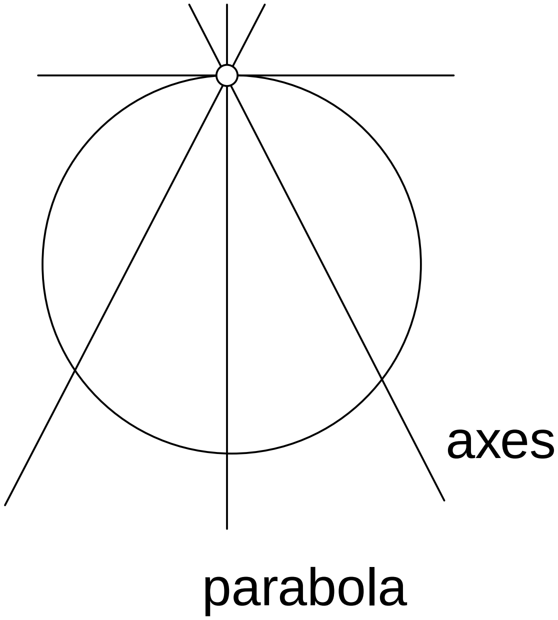
Conic sections: affine classification



... is governed by its position with respect to the line at infinity.

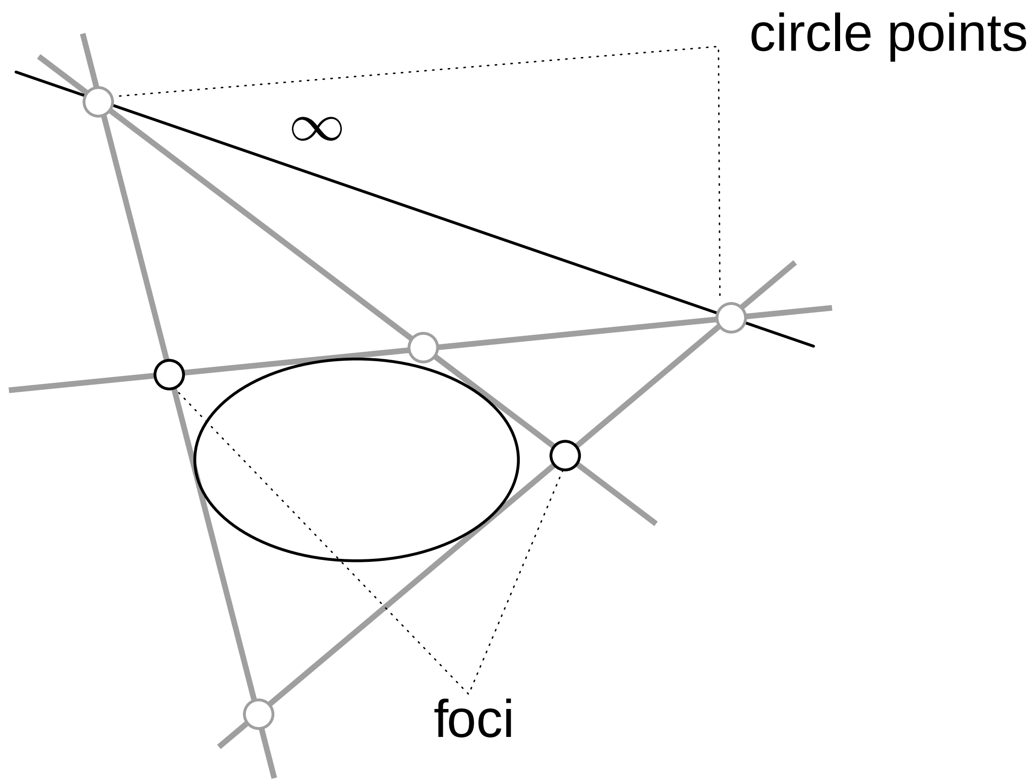
Conic sections: affine invariants





The center of an ellipse or hyperbola and the axes direction of a parabola are **affine invariants.**

Conic sections: Euclidean invariants



The **foci** of an ellipse / hyperbola are **Euclidean invariants.**

Bézier curves

- Polynomial case and affine geometry

- Rational case and projective geometry

Bézier curves: The polynomial case

- Definition by repeated convex combinations

- Definition by repeated ratios

- Properties

Definition by repeated convex combinations

Control points $\underline{\mathbf{b}}_0, \ldots, \underline{\mathbf{b}}_n$, parameter $t \in [0, 1]$

$\underline{\mathbf{b}}_0^0 = \underline{\mathbf{b}}_0$

for i from 1 to n do

    for j from 0 to $n - i$ do

        $\underline{\mathbf{b}}_j^i = (1 - t)\underline{\mathbf{b}}_j^{i-1} + t\underline{\mathbf{b}}_{j+1}^{i-1}$

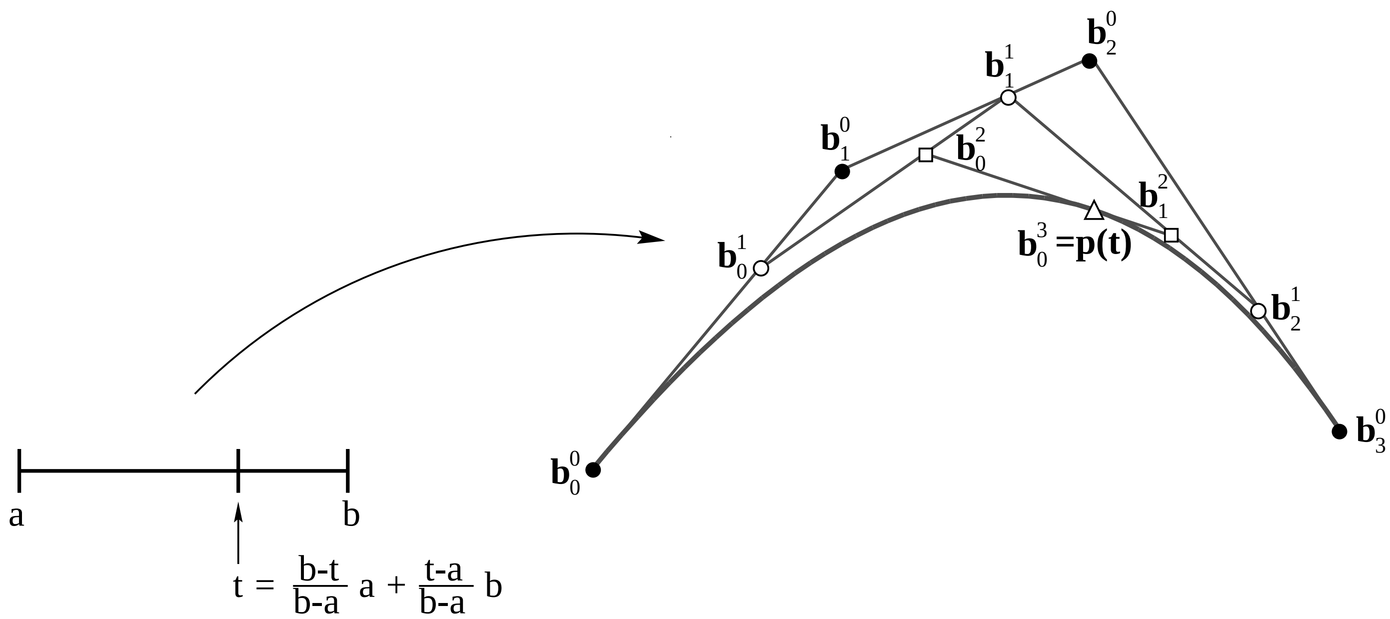
    od

od

$\underline{\mathbf{x}}(t) = \underline{\mathbf{b}}_0^n$

de Casteljau's ( ) algorithm for Bézier ( ) curves.

Definition by repeated ratios



Find $\underline{b}_i^j$ from ratio $(\underline{b}_i^j, \underline{b}_i^{j-1}, \underline{b}_{i+1}^{j-1}) = \frac{t}{1-t}$

Properties

- $\underline{\mathbf{x}}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \underline{\mathbf{b}}_i$

- All polynomial curves are Bézier curves.

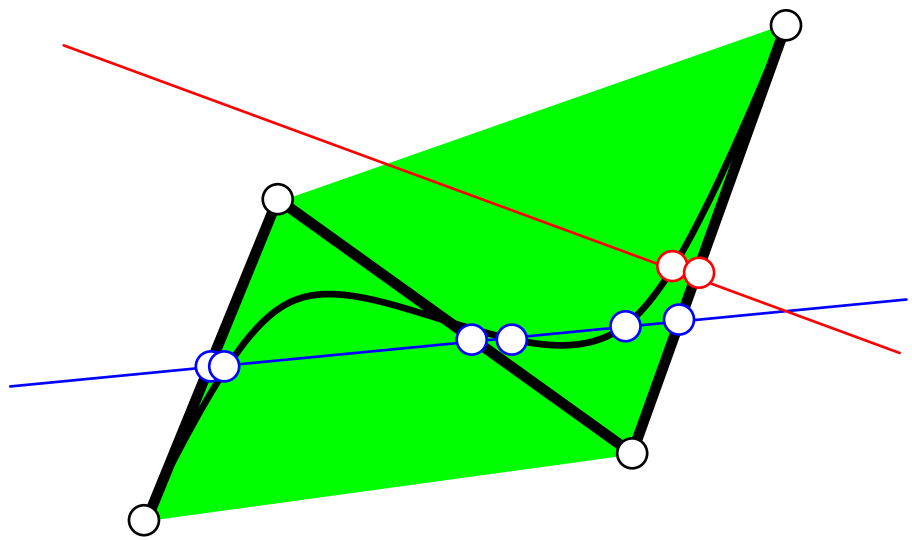
- Affine invariance:

$$\begin{array}{ccc} \{\underline{\mathbf{b}}_i\}_{i=0}^{n} & \overset{\alpha}{\rightarrow} & \{\underline{\mathbf{b}}'_i\}_{i=0}^{n} \\ \text{Evaluation} \downarrow & & \downarrow \text{Evaluation} \\ \underline{\mathbf{x}}(t) & \underset{\alpha}{\rightarrow} & \underline{\mathbf{x}}'(t) \end{array}$$

- Convex hull property

- Variation diminishing property

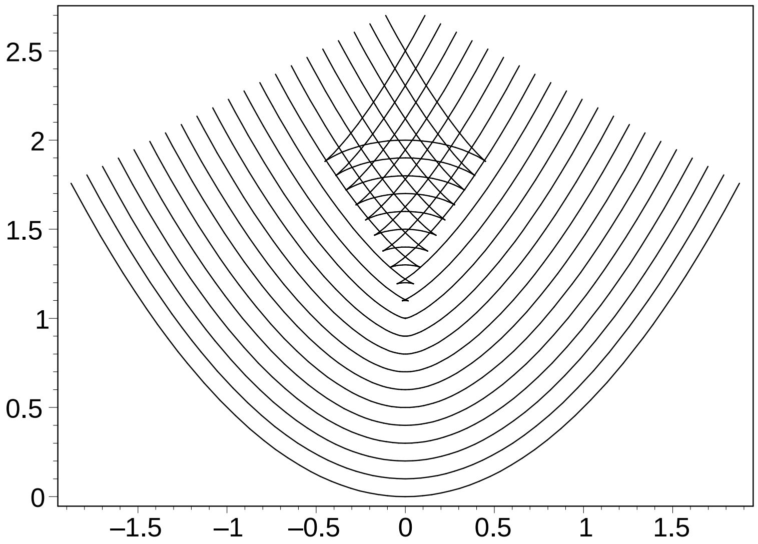
$$\#\{\underline{\mathbf{x}}(t) \cap g\} \leq \#\{[\underline{\mathbf{b}}_i]_{i=0}^{n} \wedge g\}$$



Limitations

Polynomial curves

- cannot describe conic sections (except parabolas)
- cannot represent offset curves (except offsets to straight lines)



Bézier curves: The rational case

- Definition by repeated linear combinations
- Definition by repeated cross ratios
- Properties

Definition by repeated linear combinations

Control points $\tilde{\mathbf{b}}_0, \ldots, \tilde{\mathbf{b}}_n$, Parameter $t \in [0, 1]$

$\tilde{\mathbf{b}}_0^0 = \tilde{\mathbf{b}}_0$

for i from 1 to n do
    for j from 0 to $n - i$ do
        $\tilde{\mathbf{b}}_j^i = (1 - t)\tilde{\mathbf{b}}_j^{i-1} + t\tilde{\mathbf{b}}_{j+1}^{i-1}$
    od
od

$\tilde{\mathbf{x}}(t) = \tilde{\mathbf{b}}_0^n$

A Problem ?

The result depends not only on the points, but also on the choice of the homogeneous coordinates. More precisely, a **renormalization**

$$\tilde{\mathbf{b}}_i \rightarrow \lambda_i \tilde{\mathbf{b}}_i$$

changes the curve !

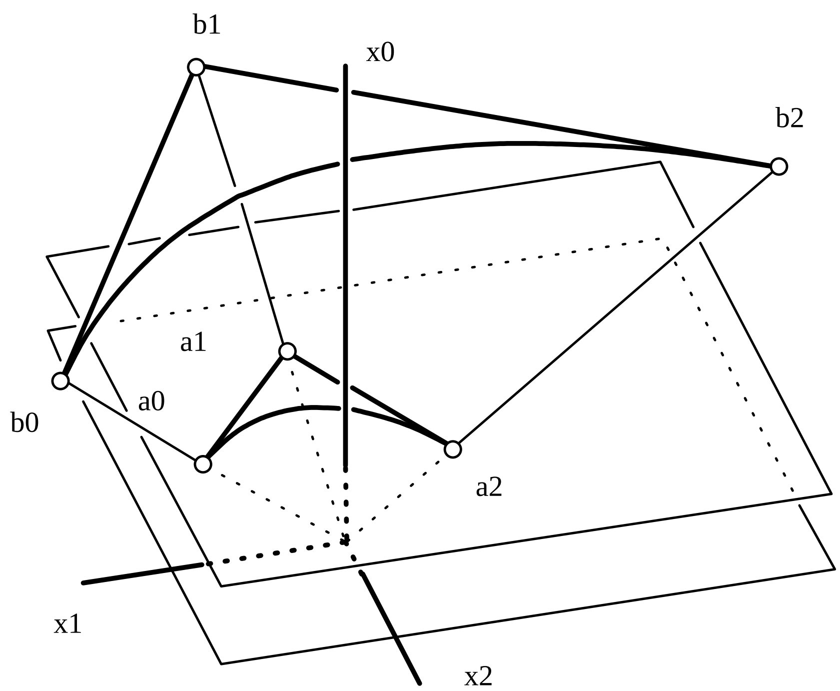
A common notation

$$\tilde{\mathbf{b}}_i = (w_i, w_i x_i, w_i y_i)^\top$$

(x_i, y_i) Cartesian coordinates, w_i “weight”

Interpretation in $\mathbb{R}^{d+1}$

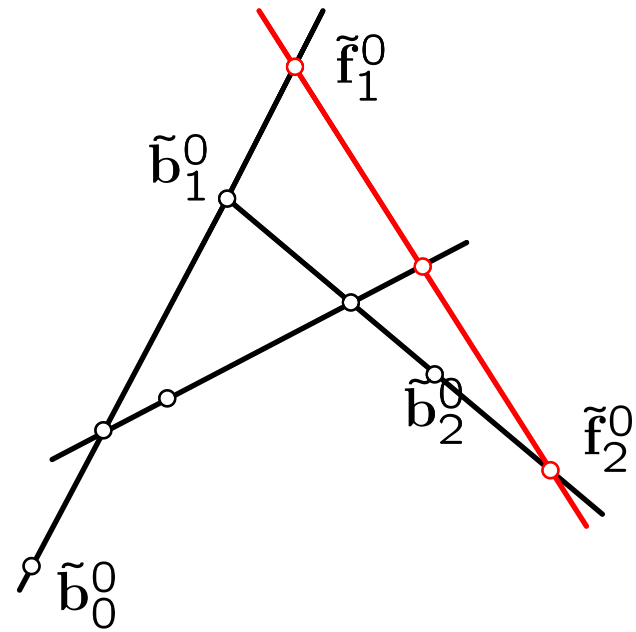
Polynomial curve in $\mathbb{R}^{d+1}$ $\circ$ central projection



Definition by repeated cross ratios

Define auxiliary points $\tilde{\mathrm{f}}_i^0 = \tilde{\mathrm{b}}_i^0 - \tilde{\mathrm{b}}_{i+1}^0$

- Find $\tilde{\mathrm{b}}_i^j$ from $\mathrm{cr}(\tilde{\mathrm{b}}_i^j, \tilde{\mathrm{f}}_i^{j-1}, \tilde{\mathrm{b}}_i^{j-1}, \tilde{\mathrm{b}}_{i+1}^{j-1}) = \frac{t}{t-1}$
- $\tilde{\mathrm{f}}_i^j = (\tilde{\mathrm{f}}_i^{j-1} \vee \tilde{\mathrm{f}}_{i+1}^{j-1}) \wedge (\tilde{\mathrm{b}}_i^j \vee \tilde{\mathrm{b}}_{i+1}^j)$



Properties

- $\tilde{\mathbf{x}}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \tilde{\mathbf{b}}_i$
- All rational curves are rational Bézier curves (J.C.F. Haase 1869)

Zur Theorie der ebenen Curven n^{ter} Ordnung mit $\frac{(n-1)(n-2)}{2}$ Doppel- und Rückkehrpunkten.

Von J. C. F. Haase in München.

In der vorliegenden Abhandlung ist ein Gegenstand berührt, dem zuerst Herr **Salmon** in seiner „*Treatise on the higher plane curves*" (p. 94) seine Aufmerksamkeit gewidmet und den sodann Herr **Clebsch** im 64. Bande des Crelle'schen Journals (pag. 43) eingehender untersucht hat. Derselbe betrifft diejenigen ebenen Curven, deren Coordinaten sich als rationale Functionen eines variabeln Parameters darstellen lassen. Während in der Arbeit des Herrn **Salmon** dieser

Hieraus ergiebt sich nun folgende Regel zur Construction *eines* Punktes einer Curve n^{ter} Ordnung mit $\frac{(n-1)(n-2)}{2}$ Doppel- und Rückkehrpunkten, für den Fall, dass 1) die Orientirungspunkte und 2) auf den Verbindungsgeraden je zweier aufeinander folgenden Orientirungspunkte entsprechende Punkte gegeben sind.

Man nehme auf einer der Verbindungsgeraden zweier aufeinanderfolgenden Orientirungspunkte $(0, 1)$, $(1, 2)$, ... $(n-1, n)$ einen beliebigen Punkt an und bestimme sodann auf allen übrigen die ihm projectivisch entsprechenden. Diese Punkte seien $0'$, $1'$, ... $(n-1)'$. Auf ihren Verbindungsgeraden $(0', 1')$, $(1', 2')$, ... $((n-2)', (n-1)')$ suche man sodann (mit Hülfe des oben citirten Brianchon'schen Satzes) diejenigen Punkte, welche demselben Werthe von λ entsprechen. Man kommt so zu neuen Punkten: $0''$, $1''$, ... $(n-2)''$. Auf den Verbindungsgeraden dieser Punkte suche man wieder die Punkte λ u. s. w. So kommt man schliesslich zu Punkten $0^{(n-2)}$, $1^{(n-2)}$, $2^{(n-2)}$. Bestimmt man auf deren Verbindungsgeraden den Punkt λ und sucht sodann endlich auf den Verbindungsgeraden der so gefundenen Punkte $0^{(n-3)}$, $1^{(n-3)}$ wieder den Punkt λ, so ist dieser ein Punkt unserer Curve.*)

Properties

- If all auxiliary points are at infinity: All cross ratios become ratios, and the rational curve becomes a polynomial one.

- Projective invariance

$$\{\tilde{\mathbf{b}}_i\}_{i=0}^{n}, \{\tilde{\mathbf{f}}_i\}_{i=0}^{n-1} \xrightarrow{\pi} \{\tilde{\mathbf{b}}'_i\}_{i=0}^{n}, \{\tilde{\mathbf{f}}_i\}_{i=0}^{n-1}$$

$$\text{Evaluation} \downarrow \qquad\qquad \downarrow \text{Evaluation}$$

$$\tilde{\mathbf{x}}(t) \quad \underset{\pi}{\longrightarrow} \quad \tilde{\mathbf{x}}'(t)$$

- Convex hull property, variation diminishing property: as before, provided that the weights are all positive.

Summary and Outlook

The hierarchy of geometries provides a useful framework for CAGD.

Non-Euclidean geometries (elliptic, pseudo-Euclidean) are sub-geometries of projective geometries. Bézier curves correspond to other objects (motions, canal surfaces).

More information on curve / surface representation will follow tomorrow.