0.0.1 Report of project part 5

Title: Computational Geometry

Principal investigator: Univ.-Doz. Dipl.Ing. Dr.techn. Oswin Aichholzer
Institute for Software Technology
Graz University of Technology
Phone/Fax: +43 316/873-5725 / -5706
URL: `http://www.ist.tugraz.at/staff/aichholzer/`
E-mail: `oaich@ist.tugraz.at`

Co–investigator(s): Univ.-Prof. Dr. Franz Aurenhammer (Graz)
Univ.-Prof. Dr. Bert Jüttler (Linz)
o.Univ.-Prof. Dr. Helmut Pottmann (Wien)

Summary

During the first three years of this NRN we have been able to obtain a large variety of results for the different proposed activities. We are proud that 17 journal articles and 18 contributions at international conferences have emerged from our NRN related research. In particular, successful cooperations with other subprojects have been established, and our Graz Computational Geometry group has been strengthened.

As highlights of the work done within this subpjoject we consider the following three core activities: The synergies of approximation theory and Computational Geometry in the context of advanced geometric representations (Activity 5.5) in cooperation with the Linz group, the sphere conversion method for triangulated 3D objects in the context of Activity 5.3 in collaboration with the Wien group, and the progress on pseudo-triangulations and their relatives (in 2D and 3D) within Activity 5.1.

Various other results, fitting into other activities, have been obtained, some still preliminary, as for example an implementation of new method for multivariate density estimation, in cooperation with the Innsbruck group.

Scientific Background / State of the art

Computational Geometry has developed a remarkable variety of efficient geometric data structures and algorithms. Still many open questions exist and work has to be done to generalize recently obtained concepts or to develop tailor-made solutions for particular applications. During the first three years of this NRN we concentrated on the following topics:

Triangulations, pseudo-triangulations and relatives: Pseudo-triangulations are a relaxation of triangulations proposed quite recently; see e.g. [AABK03] for a short bibliography. A pseudo-triangulation is a mesh composed of triangles and pseudo-triangles (polygons with exactly three convex vertices). These structures are more flexible than triangulations and have various applications. Though many of their structural and algorithmic properties have been found, several relevant questions are not well understood yet. Section 0.0.1 describes our progress in this direction.

Spheres and Voronoi diagrams: Representing a complex geometric object with primitives is a fundamental task in Computational Geometry and computer graphics. Approximations by sets of spheres are particularly suitable for further processing. Motivation comes, among other topics, from collision detection and fast computation of Minkowski sums of solids. Calculating the exact Minkowski sum of two nonconvex 3D objects is a complicated task with various applications to problems where proximity is involved [LM04] (for instance, motion planning). Present sphere conversion algorithms give guarantees on the quality but not on the minimality of the produced sets of spheres. We succeeded in producing almost-minimal sets of spheres; see Section 0.0.1. .

Geometric graphs: This topic is related to the work on pseudo-triangulations and one of our goals here was to generalize recent concepts obtained for pseudo-triangulations to general graphs. Several structural and geometric results, like angle constraints or pointedness, stayed unexplored for many other classes of geometric graphs. This also includes different variations of displaying a graph (graph drawing) or optimizing it. See Section 0.0.1 for results we have obtained during the NRN. Geometric graphs also play an implicit role in other investigated questions, for example concerning Voronoi diagrams (see previous paragraph) or medial axes (see next paragraph).

Advanced geometric representations: The majority of algorithms in Computational Geometry have been designed for processing linear objects, like lines, planes, or polygons. Curved objects are usually approximated in a piecewise-linear manner and up to a tolerable error. (Approaches to directly extending polygonal algorithms to curved objects are rare and, due to their generality, of limited practical use.) The number of line segments required for an accurate approximation may, however, be prohibitively large. Even more seriously, making a piecewise-linear approximation and invoking a polygonal-shape algorithm may generate results that are topologically incorrect. We highlight the use of circular arc approximation, which leads to significantly smaller inputs and allows the adaption and/or the new design of algorithms for basic and frequent tasks, including construction of the convex hulls, decomposition into primitives, and calculation of medial axes.

Results and Discussion

Triangulations, pseudo-triangulations and relatives: In the paper [P5-J21] we investigate the diameter of the flip graph of bounded vertex degree (and of bounded face degree) pseudo-triangulations. Bounded degree pseudo-triangulations are of special interest for applied fields, as high vertex (face) degree usually limits their practical usefulness. As a main result we show that any two (pseudo-)triangulations can be transformed into each other by a local and constant size operation (a flip), without violating a given degree bound during the whole transformation. We also give a quadratic upper bound on the number of necessary local steps. Related to these results is the question for upper and lower bounds on how many pseudo-triangulations exist for given point sets. Here we provide new relations in [P5-J13].

It is known that, given two triangulations on the same point set, a perfect matching exists between their sets of edges such that matched edges are either identical or crossing [AAC+96]. A similar property holds for their sets of triangles. We show that these results can be extended to more general polygonal partitions (including pseudo-triangulations of small rank), if vertex incidence of edges (or of faces) is considered [P5-B02, P5-J09]. To this end, we introduce general Laman (count) conditions for edges and faces in polygonal partitions. We also describe a link to spanning tree decompositions that applies to certain pseudo-triangulations and quadrangulations. A related question from *The Open Problems Project* [DMO] (Problem 50) asks whether any triangulation of a point set contains a pointed spanning tree as a subgraph. A positive result would imply an improvement of the above mentioned bounds on the flip distance for pseudo-triangulations, but in [P5-J11] we answer it in the negative.

As another partitioning-type result, we give a generalized version of the decomposition, partition and covering problem. Here we generalized the classic problems for convex sets to pseudo-triangles and convex polygons [P5-B13, P5-J12]. These novel subdivisions of the plane are significantly sparser than their convex counterparts and are thus relevant for practical applications.

Recently it has been shown that the problem of computing the minimum weight triangulation is NP-complete [MR06]. For the similar question for minimum weight pseudo-triangulations, we have been able to obtain preliminary results [P5-B06]. For example, minimum weight pseudo-triangulations need not to be pointed, and consequently, they need not have the minimum number of edges among all possible pseudo-triangulations of the given point set. Moreover, the greedy pseudo-triangulation always turns out to be a full triangulation, a result that shows that this concept is not meaningful. The main problem of resolving the complexity status of minimum weight pseudo-triangulations remains open.

As a major new concept, we generalize triangulations in a natural way beyond pseudo-triangulations, in the work [P5-I05, P5-B03, P5-J10]. A pseudo-triangle is a simply connected polygonal region where exactly three vertices have no reflex angle. Dropping simplicity, we arrive at a concept we termed a pre-triangle, and following suit, a pre-triangulation of a given polygonal domain. We show that pre-triangulations arise in three different contexts: In the characterization of complexes that are liftable to three-space in a strong sense, in flip sequences for general polygonal complexes, and as graphs of maximal locally convex functions and optimal surfaces.

Among the most challenging tasks in this context is the generalization of pseudo-triangulations to higher-dimensional space. We define, in the paper [P5-J08], pseudo-simplices and pseudo-simplicial complexes in d-space in a way consistent to pseudo-triangulations in the plane. Flip operations in pseudo-complexes are specified as combinations of flips in pseudo-triangulations and of bistellar flips in simplicial complexes. A certain class of pseudo-complexes is shown to be connected under such flips. This class also admits a representation as a high-dimensional convex polytope, which generalizes known polytope constructions for regular triangulations and pseudo-triangulations. We thus unify several well-known structures, including pseudo-triangulations, constrained Delaunay triangulations [Che89, She03], and regular simplicial complexes [Lee91, ES96].

Spheres and Voronoi diagrams: In collaboration with the Vienna group around Prof. Pottmann we investigated the approximation of three-dimensional solids with a reasonable small number of balls [P5-B05]. We developed an algorithm that takes as input a 3D object with triangulated boundary, and generates an almost-minimal set of balls that covers all the triangle endpoints, ensuring that no such point is covered with more than a user-specified offset ε. The quality of the approximation thus will also depend on the quality of the boundary mesh. If the object boundary is not covered completely, this can be achieved if desired, with a simple postprocessing step. Following known paths [AB99, ACK01], we first generate a candidate set of approximating balls centered on the triangles endpoints' Voronoi diagram. Correct labeling of balls as having their centers inside or outside the object becomes an issue, and we propose a simple though efficient labeling algorithm using the boundary triangulation. A method for reducing the candidate set of balls is then applied as an instance of the set covering problem. The combination of exact and heuristic methods we use allows us to determine how close to the optimum the produced set of balls is.

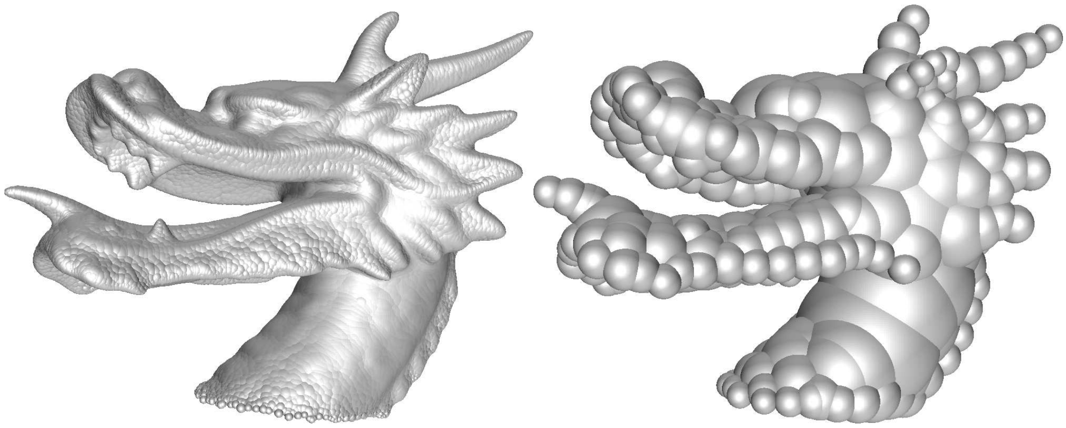

Figure 1: Benchmark dragon converted into spheres.

Test results showed that, within a given geometric error bound, we can approximate given solids with a significantly smaller number of balls than previous approaches (like e.g. the power crust approach [ACK01]). Our experiments, including the benchmark example in Figure 1, also show very satisfactory results with respect to preserving the topology. The overhead in balls (compared to the optimal solution) is negligible. We also have obtained a stable implementation of our algorithm using the CGAL library [CGA].

Having available two solids converted into approximating sets of balls, their approximate Minkowski sum can be calculated very easily; computing the Minkowski sum of two balls is a trivial task. We have implemented our approximate Minkowski sum algorithm using CGAL [CGA]. The main objective is to avoid considering balls not contributing to the solution. This problem can be solved using the power diagram of the balls: A given ball contributes to the union of a set of balls with exactly those parts contained in its power cell. Preliminary tests showed that the set of balls constituting the Minkowski sum can be pruned drastically with this approach. A rapid and relatively accurate computation of Minkowski sums for general 3D shapes is within reach in this way.

An interesting 2D application of our ball conversion algorithm has been investigated in collaboration with the Institute of Mechanics and the Institute of Geometry of Graz University of Technology. Here, for the analysis of planar bar mechanisms, the error workspace is approximated by disks. This allows for an efficient computation of a combination of several error workspaces.

We finally mention that, as a byproduct, the ball conversion algorithm yields a basis for computing an approximate medial axis of the input object: Computing the exact medial axis of a finite union of balls is a well-understood task [AK01]. In view of the lack of satisfactory 3D medial axis algorithms, and the (almost) optimality of our ball conversion algorithm, we plan to investigate this topic further.

Another collaboration with Subproject 06, within our Activity 5.3, concerns Voronoi diagrams for oriented spheres, or, equivalently, points in the quasi-Euclidean space [P5-B18]. This work is motivated by special relativity theory, where one describes events as points in quasi-Euclidean space. The respective distance, d, between two points can be time-like, or space-like, or light-like, depending on whether the sign of d is negative, positive, or zero. We investigated several variants of the quasi-Euclidean distance and their corresponding Voronoi diagrams – with respect to space requirement and construction time.

As a related work, we investigated the farthest line segment Voronoi diagram in the Euclidean plane. This type of diagram, surprisingly, has been treated as a stepchild in Computational Geometry. In [P5-J07], we give a complete combinatorial analysis and an optimal construction algorithm for farthest line segment Voronoi diagrams.

Another work done within Activity 5.3 deals with fixed-share decompositions of convex polygons into convex parts [P5-J16]. Share may relate to area covered, or to number of points cut off from a given initial set. The so-called weighted skeleton of a polygon (a weighted version of straight skeletons [AAAG95]) is introduced, and various respective decomposition and optimality results for this skeletal structure are presented.

In a collaboration with the Innsbruck group around Prof. Scherzer, we have been developing a method for estimating density functions on multivariate data [P5-J20], generalizing their approach for bivariate data [OSK07]. The planar method, first having been based on the Delaunay triangulation of the data points, has been modified to use (sub)areas of Voronoi regions, and is extended to higher dimensions. Literature on higher-dimensional density estimation is sparse.

For an efficient and effective implementation of our approach, we need an efficient calculation of high-dimensional Voronoi diagram subcells, and a method for pruning the data before applying the Voronoi diagram approach. While efficient high-dimensional Voronoi diagram algorithms are available from the CGAL library [CGA], the computation of the subcell volumes has still to be implemented. Moreover, and equally important, an appropriate method has to be found for clustering the input data so that essential features are preserved. We plan to use, among other methods, the complete and average linkage clustering methods and their variants, as developed in [Aic96].

Geometric graphs: Several results have been obtained concerning geometric graphs (Activity 5.4). Many of them are strongly related to the special class of pseudo-triangulations (Activity 5.1), see the paragraph about pseudo-triangulations above. Thus below we mention only results not (directly) related to pseudo-triangulations.

Epsilon nets, which play an important role e.g. in the design of efficient randomized algorithms for geometric partition problems (trapezoidations) and intersection problems (intersection of line segments), have been investigated in [P5-B17]. A variety of combinatorial results which are related to the complete geometric graph are presented in [P5-J03, P5-B14, P5-B15]. A recent result for efficiently generating different families of plane graphs (including for example spanning trees) with the help of Gray codes has been developed in [P5-B07, P5-J14, P5-T02]. This is related to a very general result showing tight lower bounds on the number of such structures [P5-B11, P5-B12, P5-J04, P5-T02]. Moreover, for skeletal structures, like e.g. trees, we have developed a fast way of transforming them into each other [P5-J02], even if single steps in the transformation are limited to very basic slide operations [P5-J06].

In [P5-J01] we show a new Ramsey-type result: For a two-colored point set we give an $O(n \log n)$ time algorithm to construct two crossing-free spanning trees, one for each color, where the trees are optimized w.r.t. their diameters and their pairwise intersections.

A classical question for geometric graphs is concerned with so-called k-sets (k-edges). Here the structure of point sets minimizing the number of k-sets or $(\leq k)$-sets is a long-standing open problem in combinatorial graph theory. We extended the work of [P5-B15] to [P5-J15] and gave new bounds and novel structural relations of optimal sets. The results provide improved asymptotic bounds on the rectilinear crossing number as well as concrete values for small sets.

The latter question has also led to an interesting project on using world-wide distributed computing for the determination of the rectilinear crossing number. The success of the so-called RCN-project was even noted in the popular german newspaper 'Frankfurter Allgemeine Zeitung' (see 'Natur und Wissenschaft' from February 7th, 2007). In [P5-B08, P5-J19] tightness of several of the obtained results has been shown and first steps of the generalization to 3D have been obtained. Related results have also been proven in [P5-J05] and [P5-J03].

The reflexivity of a point set S is the minimum number of reflex vertices in a simple polygonalization (spanning Hamiltonian cycle) of S. This notion was suggested by Arkin et al. as a measure for the "goodness" of a polygonalization of S, and providing good bounds is among the prominent questions listed in the in *The Open Problems Project* [DMO] (Problem 66). In a recent paper [P5-B01] we present the currently best bounds for this measure.

In [P5-B10, P5-B09] and [P5-T02] we generalized the concept of pointed pseudo-triangulations (see Activity 5.1) to geometric graphs, where we guarantee for each point an incident angle not smaller than a predefined value. We provide tight lower angle-bounds for triangulations, spanning trees, spanning trees with bounded vertex degree, and spanning paths.

In a similar flavour we have shown that any planar graph can be drawn (embedded) in a pointed way, using simple curves like tangent-continuous biarcs or quadratic Bèzier curves (parabolic arcs) [P5-B16]. That is, for any vertex we can guarantee an incident halfspace within no edges emanate from this vertex. This can for example be used to place sufficiently large labels near each vertex. Our approach even allows to choose the direction of pointedness for every vertex individually.

Advanced geometric representations: In collaboration with the Linz group (Prof. Jüttler), we studied boundary approximations of planar shapes by circular arcs, and worked out quantitive and qualitative advantages compared to using straight-line segments [P5-B04]. We demonstrate this by way of three basic and frequent computations on shapes – convex hull, decomposition, and medial axis.

We assume that the boundary of the given shape is piecewise-polynomial of constant degree, and we use biarcs (pairs of smoothly joined circular arcs) [MW99] as primitives. An approximating spline curve b of size n is computed in $O(n)$ time. It fits the input curve in slope at biarc endpoints, and can be tuned to match it in curvature at certain points (a fact being important in subsequent medial axis computations). The approximation order of b is three, in contrast to using line segments where one cannot exceed order two, and a polyline of size $\Theta(n^{3/2})$ is needed to arrive at the same precision.

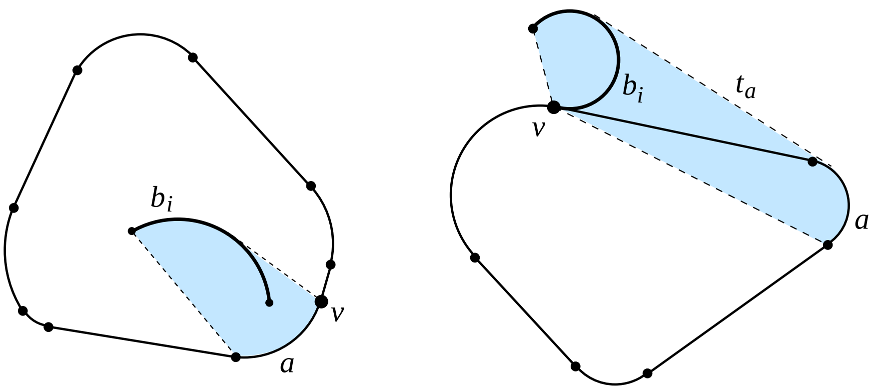

Figure 2: Incremental convex hull for circular arc shape

We then outline an algorithm for computing the convex hull of a circular boundary shape, a basic task for many subsequent computations. The incremental convex hull method by Melkman [Mel87] stands out by its simplicity, and it is this candidate we generalize for circular arc shapes. Compared to the original setting, two difficulties arise. Deciding inclusion for a currently inserted arc in the convex hull constructed so far is no trivial test, and the convex hull cannot be described by a sequence of input vertices of the shape. See Figure 2. We show that a runtime of $O(n)$ is still possible.

Our next result deals with shape triangulation, a fundamental building block in algorithms for decomposition, shortest path finding, and visibility – to name a few. Most existing algorithms are meant for polygonal shapes. When trying to generalize to shapes bounded by circular arcs, we face two problems. First of all, if the use of extraneous points is disallowed, then a partition of the shape into primitives bounded by a constant number of circular arcs need not exist. Also, not all triangulation methods are suited to generalization. The triangulation algorithm we propose is closest to Chazelle's [Cha82]. It manages with an (almost) worst-case minimal number of Steiner points on the boundary, runs in $O(n \log n)$ time, and uses a dictionary as its only nontrivial data structure. The most complex geometric operation is intersecting a circle with a line.

Finally, and maybe most important, we consider the medial axis. We present a simple randomized divide-and-conquer algorithm that overcomes the drawbacks of previous approaches. In contrast to comparable algorithms, the costly part is delegated to the divide step. The basic subroutine there is an inclusion test for an arc in a circle. The merge step is trivial: it concatenates two medial axes. The expected runtime is bounded by $O(n^{3/2})$, but is provably better for most types of shape. For example, $O(n \log n)$ expected time suffices if the diameter of the medial is $\Theta(n)$ (a quite realistic assumption if the shape is approximated accurately). No nontrivial data structures are used by the algorithm. We point out that, even for shapes most unfavorable to our algorithm, the expected runtime is linear in the quantity $N = n^{3/2}$ describing the size of an approximating polyline.

To guarantee applicability of our methods to approximating the medial axes of general shapes A, we prove that, for a suitable approximation of ∂A by biarcs, $M(A)$ is the limit of $M(B)$ when the approximating arc shape B converges to A. Related results exist, but either presuppose C^2 conditions on ∂B not attainable by circular arcs [CS04], or concern subsets of the medial axis [CL04] that survive after pruning the Voronoi diagram of point samples from ∂A. (As a negative side effect, the medial axis approximation obtained from a point sample is not C^1.) It is well known [ABE07] that medial axis convergence is *not* given for polygonal approximations of A. In conclusion, circular arcs are the simplest possible tool for boundary conversion that guarantees a stable medial axis approximation.

National and international cooperation

- Work on several variations of Voronoi diagrams was done during the research visits of B. Balop del Rio (Minimum Manhattan networks), T. Asano (angular and aspect ratio Voronoi diagrams), and R.L.S. Drysdale (aspect ratio Voronoi diagrams).

- With M. Teillaud we work on the integration of our algorithms into the Computational Geometry Algorithms Library (CGAL). This cooperation extended to a bilateral project 'Amadée'.

- During the visit of M. van Kreveld and R.I. Silveira work on surface triangulation and optimizing flips in triangulations was initiated, and during the stays of C. Huemer several results concerning geometric graphs have been obtained [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**][1]. Together with R. Fabila-Monroy and D. Flores-Penaloza these investigations

[1]These citations refer to the CV of O.Aichholzer, see page **??**.

have been continued.

- In cooperation with M. Hoffmann and B. Speckmann we worked on several problems on pseudo-triangulations and related structures [?, ?, ?, ?][1], gaining the main results at research stays at TU-Graz and TU Eindhoven.

- Other cooperations include work on the number of crossings of geometric graphs (with P. Ramos and D. Orden) [?, ?][1], questions on pseudo-triangulations and their generalization to geometric graphs (F. Hurtado, F. Santos, S. Bereg, G. Rote, and A. Schulz) [?, ?, ?, ?, ?, ?, ?][1], and combinatorial motion planing for micro-robots (V. Sacristan).

- Several talks on subjects from Computational Geometry have been given by visitors during the first period of the NRN, for example Bernard Chazelle (Princeton University): Data-Powered Geometric Computing, Erik Demaine (Massachusetts Institute of Technology): Computational Geometry through the Information Lens, Herbert Edelsbrunner (Duke University): Introduction to persistent homology, and Günter Rote (FU Berlin): Berechnung der kürzesten Triangulierung ist NP-schwer (about the NP-hardness result for MWTs), to just name the most prominent speakers.

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Invited talks

[P5-I01] O. Aichholzer. Points and Combinatorics. Meeting of AMS,DMV,MG, Mainz (Germany), 2005 (invited plenary lecture).

[P5-I02] O. Aichholzer. Maximizing maximal angles for plain straight-line graphs. 6^{th} Slovenian International Conference on Graph Theory, Bled (Slovenia), 2007.

[P5-I03] O. Aichholzer. On the Number of Planar Geometric Graphs. Research visit Universidad de Valladolid, Prof. B. Palop, Valencia (Spain), 2005.

[P5-I04] O. Aichholzer and B. Vogtenhuber. Maximizing Maximal Angles for Plane Straight-Line Graphs. INRIA Sophia Antipolis Geometrica Seminar, Antibes (France), Juli 2007.

[P5-I05] F. Aurenhammer. Pre-triangulations: A generalization of Delaunay triangulations and flips. 2^{nd} Intern. Symp. on Voronoi Diagrams in Science and Engineering, Hanyang University, Seoul, Korea, 2005 (invited plenary lecture).

[P5-I06] F. Aurenhammer. Voronoi diagrams for oriented spheres. Dagstuhl Seminar 07111: Computational Geometry, Wadern (Germany), 2007.

[P5-I07] T. Hackl. Maximizing Maximal Angles for Plane Straight Line Graphs. Dagstuhl Seminar 07111: Computational Geometry, Wadern (Germany), 2007.

[P5-I08] B. Kornberger. Approximation of solids by unions of spheres. INRIA Sophia Antipolis Geometrica Seminar, Antibes (France), December 2006.

Contributed talks and poster presentations

[P5-C01] O. Aichholzer. On the Number of Planar Geometric Graphs. Second European Workshop on Pseudo-Triangulations, Alcalá de Henares (Spain), 2005.

[P5-C02] O. Aichholzer. Abstract Order Type Extension and New Results on the Rectilinear Crossing Number. 21^{st} Annual ACM Symposium on Computational Geometry (SoCG), Pisa (Italy), 2005.

[P5-C03] O. Aichholzer. On the Number of Plane Graphs. 17^{th} Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), Miami (Florida, USA), 2006.

[P5-C04] O. Aichholzer. Poster presentation: *New lower bounds for the number of* $\leq k$*-edges and the rectilinear crossing number of* K_n. International Congress of Mathematicians, Madrid (Spain), 2006. (First prize poster competition, Section "Combinatorics").

[P5-C05] F. Aurenhammer. Pseudotetraeder-Netze. Workshop "Inverse Problems", Obergurgl (Austria), 2005.

[P5-C06] F. Aurenhammer. Pre-triangulations and liftable complexes. 22^{nd} Annual ACM Symposium on Computational Geometry (SoCG), Sedona (USA), 2006.

[P5-C07] F. Aurenhammer. Voronoi diagrams for oriented spheres. Intern. Symposium on Voronoi Diagrams, Cardiff (UK), 2007.

[P5-C08] F. Aurenhammer. A practical 2D medial axis algorithm. Computational Geometry Days, Taipei (Taiwan), 2007.

[P5-C09] T. Hackl. Matching edges and faces in polygonal partitions. 17^{rd} Canadian Conference on Computational Geometry, Windsor (Ontario, Canada), 2005.

[P5-C10] B. Kornberger. Approximation of solids by unions of spheres. ACS General Workshop, Berlin (Germany), 2007.

[P5-C11] B. Kornberger. Approximating Boundary-Triangulated Objects with Balls. 23^{rd} European Workshop on Computational Geometry, Graz (Austria), 2007.

[P5-C12] B. Vogtenhuber. Gray code enumeration of plane straight-line graphs. 22^{nd} European Workshop on Computational Geometry, Delphi (Greece), 2006.

[P5-C13] B. Vogtenhuber. Maximizing Maximal Angles for Plane Straight Line Graphs. 23^{rd} European Workshop on Computational Geometry, Graz (Austria), 2007.

[P5-C14] B. Vogtenhuber. Maximizing Maximal Angles for Plane Straight Line Graphs. 10^{th} International Workshop on Algorithms and Data Structures (WADS), Halifax (Nova Scotia, Canada), 2007.

[P5-C15] B. Vogtenhuber. Pointed Drawings of Planar Graphs. 19^{th} Canadian Conference on Computational Geometry, Ottawa (Ontario, Canada), 2007.

Technical reports and theses

[P5-T01] W. Aigner. The Medial Axis of Planar Shapes. *Master's thesis, IGI-TU Graz*, Austria, September 2007. Supervisor: Franz Aurenhammer.

[P5-T02] B. Vogtenhuber. On Plane Straight-Line Graphs. *Master's thesis, IST-TU Graz*, Austria, January 2007. Supervisor: Oswin Aichholzer.

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